Lecture 14: Recall :

Definition: Let T be a linear operator on a vector space V.

A subspace $W \subset V$ is called T-invariant if $T(W) \subseteq W$.

That is, $T(\vec{w}) \in W$ for $\forall \vec{w} \in W$.

Def: Given a linear operator T on a vector space V, and a non-zero $\vec{x} \in V$, the subspace

$$W := \operatorname{span}\left(\{ T^k(\vec{x}) : k \in \mathbb{N} \}\right) \overset{\text{def}}{=} \operatorname{span}\left(\{ \vec{x}, T(\vec{x}), T^2(\vec{x}), \dots, T^k(\vec{x}), \dots \}\right)$$

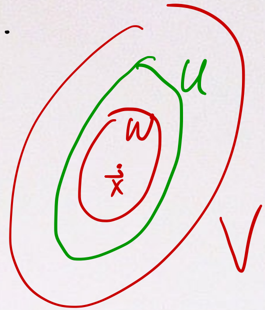
$$\left(T^k \overset{\text{def}}{=} \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}} \right)$$

is called T- cyclic subspace of V generated by $\vec{x}$.

Prop: W is the smallest T-invariant subspace of V containing $\vec{x}$.

Proof: For any $\vec{w} \in W$, $\exists\, a_0, \dots, a_k \in F$ s.t.

$$\vec{w} = \sum_{i=0}^{k} a_i T^i(\vec{x})$$



Then: $$T(\vec{w}) = \sum_{i=0}^{k} a_i T^{i+1}(\vec{x}) \in W.$$

$\therefore$ W is T-invariant.

If $U \subset V$ is a T-invariant subspace containing $\vec{x}$.

then: it also contains $T(\vec{x})$ and $T^k(\vec{x})$ by induction.

$\in U$ $\qquad$ $\in U$

$\therefore\ U \supset W$

Example: • For $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f(x)) = f'(x)$

then T-cyclic subspace generated by x^n is:

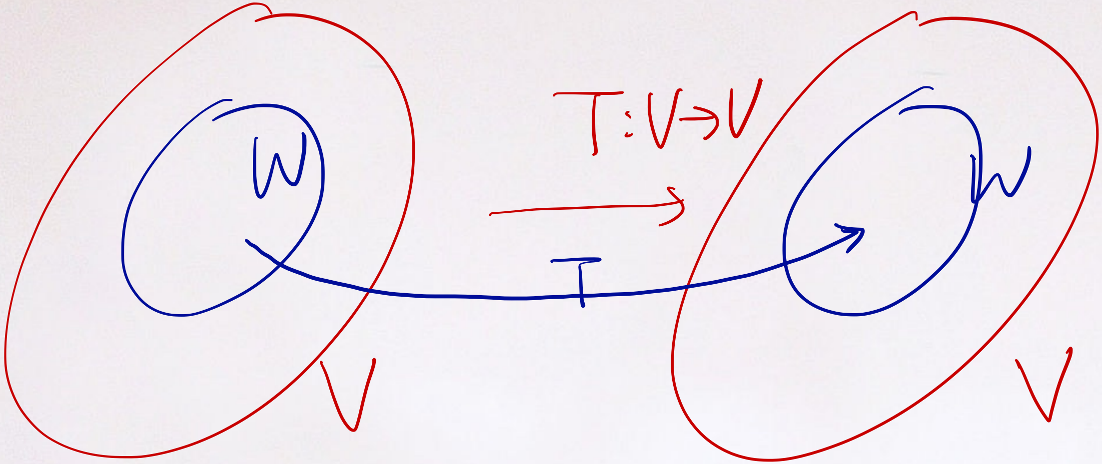
$$\text{span}\left(\{x^n, n x^{n-1}, \ldots, n!\,x, n!\}\right) = P_n(\mathbb{R})$$

• Let $T: V \to V$ be linear. Then, a 1-dimensional T-invariant subspace $U \subset V$ is nothing but the span of an eigenvector of T.

[If U = 1-dim T-invariant subspace.
Then, $U = \text{span}\{\vec{v}\}$. Then: $T(\vec{v}) \in U$ $\Rightarrow T(\vec{v}) = \lambda \vec{v}$. $\therefore \vec{v}$ = eigenvector of T.

$\vec{v} \neq \vec{0}$

Also, if $\vec{v} \in V$ is an eigenvector of T, then T-cyclic subspace generated by $\vec{v}$ is also $\text{span}\{\vec{v}\}$. $\left(= \{\vec{v}, \underset{\lambda\vec{v}}{\cancel{T(\vec{v})}}, \underset{\lambda^2\vec{v}}{\cancel{T^2(\vec{v})}} \ldots \}\right)$



Def: $T|_W : W \to W$ defined by $T|_W(\vec{w}) = T(\vec{w})$

$f_{T|_W}(t)$ divides $f_T(t)$

Remark: Let $T: V \to V$ be a linear operator on a finite-dim vector space V, and let $W \subset V$ be a T-invariant subspace.

Then, the restriction of T to W, denote it by $T|_W : W \to W$, is well-defined and linear.

Proposition: $f_{T|_W}(t)$ divides $f_T(t)$.

Proof: Choose an ordered basis $\gamma = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ for W and extend it to an ordered basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for V. Then:

$$[T]_\beta = \begin{pmatrix} | & | & \\ [\overset{\in W}{T(\vec{v}_1)}]_\beta & [T(\vec{v}_2)]_\beta & \cdots \\ | & | & \end{pmatrix} = \begin{array}{cc} \begin{array}{cc} \overbrace{\qquad\quad}^{k} & \qquad \end{array} & \\ \begin{pmatrix} [T|_W]_\gamma & B \\ O & C \end{pmatrix} & \begin{array}{l} \}\,k \\ \end{array} \end{array}$$

$$f_T(t) = \det\left[\begin{pmatrix} [T_W]_\gamma & B \\ O & C \end{pmatrix} - t\,I\right]$$

$$= \det\begin{pmatrix} [T_W]_\gamma - t\,I_k & B \\ O & C - t\,I_{n-k} \end{pmatrix}$$

$$= \det\left([T_W]_\gamma - t\,I_k\right) \underbrace{\det\left(C - t\,I_{n-k}\right)}_{g(t)}$$

$$= f_{T_W}(t)\; g(t)$$

$\therefore$ $f_{T_W}(t)$ divides $f_T(t)$

Theorem: Let $T: V \to V$ be a linear operator on a finite-dim vector space V and let $W \subset V$ be T-cyclic subspace of V generated by $\vec{v} \neq \vec{0} \in V$. $(W = \operatorname{span}\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots\})$

Let $k = \dim(W)$. Then:

(a) $\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots, T^{k-1}(\vec{v})\}$ is a basis for W

(b) If $a_0\vec{v} + a_1 T(\vec{v}) + a_2 T^2(\vec{v}) + \dots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$, then the characteristic polynomial of $T|_W$ is:

$$f_{T_W}(t) = (-1)^k\left(a_0 + a_1 t + a_2 t^2 + \dots + a_{k-1}t^{k-1} + t^k\right)$$

<u>Proof:</u> (a) Since $\vec{v} \neq \vec{0}$, then $\{\vec{v}\}$ is linearly independent.

Let j be the largest +ve integer s.t.

$$\beta = \{\vec{v}, T(\vec{v}), \dots, T^{j-1}(\vec{v})\} \text{ is linearly independent.}$$

Such j exists because V is finite-dim.

Let $Z = \operatorname{span}(\beta)$. $\therefore$ $Z \subset W$

Then, $\underbrace{\beta}_{\text{L.I.}} \cup T^j(\vec{v})$ is linearly dependent. $\therefore$ $T^j(\vec{v}) \in \underbrace{\operatorname{span}(\beta)}_{}$ $\left(\operatorname{span}(\beta) = Z\right)$

$\therefore$ $T^j(\vec{v}) \in Z$.

Now, let $\vec{w} \in Z$. Then $\exists\ b_0, b_1, \dots, b_{j-1} \in F$ s.t.

$$\vec{w} = b_0 \vec{v} + b_1 T(\vec{v}) + \dots + b_{j-1} T^{j-1}(\vec{v})$$

$$T(\vec{w}) = b_0 \underset{\in Z}{T(\vec{v})} + b_1 \overset{\in Z}{T^2(\vec{v})} + \dots + b_{j-2} \overset{\in Z}{T^{j-1}(\vec{v})} + b_{j-1} \overset{\in Z}{T^j(\vec{v})} \in Z$$

$\therefore$ If $\vec{w} \in Z$, then $T(\vec{w}) \in Z$.

$\therefore$ Z is T-invariant subspace containing $\vec{v}$.

$\therefore$ $\underbrace{W}_{\substack{\shortparallel \\ T\text{-cyclic subspace} \\ \text{containing } \vec{v}}} \subset Z$. ($\because$ W is smallest T-invariant subspace containing $\vec{v}$)

$\therefore$ $W = Z = \operatorname{span}(\overbrace{\beta}^{\text{L.I.}})$

$\therefore$ β is a basis of W and $j = k$.

(b) By (a), $\beta = \{\vec{v}, T(\vec{v}), \dots, T^{k-1}(\vec{v})\}$ is an ordered basis for W.

Let $a_0, \dots, a_{k-1} \in F$ s.t.

$$a_0\vec{v} + a_1 T(\vec{v}) + \dots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$$

$$\Rightarrow T^k(\vec{v}) = -a_0\vec{v} - a_1 T(\vec{v}) - \dots - a_{k-1} T^{k-1}(\vec{v}).$$

Then:
$$[T]_\beta = \begin{pmatrix} | & | & & | \\ [T(\vec{v})]_\beta & [T(T(\vec{v}))]_\beta & \cdots & [T^k(\vec{v})]_\beta \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

$$f_{T_W}(t) = \det\left(\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & \ddots & 0 & \cdots 0 & -a_1 \\ & \ddots & \ddots & & \vdots \\ & & & 1 \quad 0 & -a_{k-1} \end{pmatrix} - t I_k\right)$$

$$= (-1)^k\left(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k\right) \quad \text{(HW)}$$

<u>Theorem:</u> (Cayley-Hamilton) Let T be a linear operator on a finite-dim. vector space V and let $f(t) = f_T(t)$ be a char poly of T. Then: $f(T) =$ zero transformation.

(Char poly "kills" the linear operator T)

<u>Remark:</u>

$$f(t) = a_0 1 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \dots + a_n T^n$$

Proof: We want to show $f(T)(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.

$f(T)(\vec{0}) = \vec{0}$ ($\because$ $f(T)$ is linear)

So, suppose $\vec{v} \neq \vec{0}$. Let W = T-cyclic subspace generated by $\vec{v}$.

Let $k = \dim(W)$

By Thm we have shown last time:

$\exists\ a_0, a_1, \dots, a_{k-1} \in F$ such that:

$$\begin{cases} \cdot\ a_0 \vec{v} + a_1 T(\vec{v}) + \dots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0} \\ \cdot\ g(t) \overset{\text{def}}{=} f_{T|_W} = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k) \end{cases}$$

$$\begin{cases} \cdot \ a_0 \vec{v} + a_1 T(\vec{v}) + \dots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0} \\ \cdot \ g(t) \overset{\text{def}}{=} f_{T|_W} = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k) \end{cases}$$

$$\Downarrow$$

$$g(T)(\vec{v}) = \vec{0}$$

Now, $g(t) \mid f(t) \overset{\text{implies}}{\rightsquigarrow} \exists\, q(t)$ s.t. $f(t) = q(t)\, g(t)$

$$f(T) = q(T)\, g(T)$$

$$q(T) \circ g(T)$$

$$\therefore \quad f(T)(\vec{v}) = q(T) \circ \underbrace{g(T)(\vec{v})}_{\vec{0}} = \vec{0}$$

<u>Corollary:</u> Let $A \in M_{n\times n}(F)$ and $f(t)$ be its char. poly. Then: $f(A) = O$, the zero matrix.

Inner product and norm

Assume $F = \mathbb{R}$ or $\mathbb{C}$.

Definition: Let V be a vector space over F. An inner product on V is a map $\langle \cdot , \cdot \rangle : V \times V \to F$ s.t. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $c \in F$, it satisfies:

(a) $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

(b) $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

(c) $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$

(d) $\underbrace{\langle \vec{x}, \vec{x} \rangle}_{\in \mathbb{R}} > 0$ if $\vec{x} \neq \vec{0}$

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Remark: • (a), (b) say that the inner product is linear in its argument.

• If $F = \mathbb{R}$, $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Example: • For $\vec{x} = (a_1, a_2, \dots, a_n)$, $\vec{y} = (b_1, b_2, \dots, b_n) \in F^n$
$(F = \mathbb{R}, \mathbb{C})$

We have: standard inner product

$$\langle \vec{x}, \vec{y} \rangle \overset{\text{def}}{:=} \sum_{i=1}^{n} a_i \overline{b_i}$$

• If $\langle \cdot, \cdot \rangle$ is an inner product on V, and $r > 0$,

then: $\langle \vec{x}, \vec{y} \rangle' \overset{\text{def}}{:=} r \langle \vec{x}, \vec{y} \rangle$ is another inner product on V.

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- Let $V = C([0,1])$ be vector space of real-valued continuous functions on $[0,1]$. Then: for $f, g \in V$,

$$\langle f, g \rangle \stackrel{def}{=} \int_0^1 f(t)\, g(t)\, dt$$ defines an inner product on V.

$(F = \mathbb{R}, \mathbb{C})$

- Let $V = M_{n \times n}(F)$. For $A, B \in V$, we define:

sum of diagonal entries.

$$\langle A, B \rangle \stackrel{def}{=} tr(B^* A)$$

where B^* is the conjugate transpose of B defined by:

$$B^* = \overline{B}^{\,T}$$