Lecture 12: Recall:

Eigenvalue & Eigenvectors

Def: A linear operator $T: V \to V$ (where V is finite-dim) is called diagonalizable if $\exists$ an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

A square matrix A is called diagonalizable if L_A is so.

Def: Let T be a linear operator on a vector space V/F.

A non-zero vector $\vec{v} \in V$ is called an eigenvector of T if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \vec{v}$. In this case, $\lambda \in F$ is called an eigenvalue corresponding to the eigenvector $\vec{v}$.

Prop: A linear operator $T: V \to V$ (V = fin-dim) is diagonalizable iff $\exists$ an ordered basis β for V consisting of eigenvectors of T.

In such case, if $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then:

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

where λ_j is the eigenvalue of T corresponding to $\vec{v}_j$.

Prop: Let $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Def: Let T be a linear operator on an n-dim vector space V. Choose an ordered basis β for V. Then, the characteristic polynomial of T is defined as the characteristic polynomial of $[T]_\beta$.

(i.e. $f_T(t) \overset{\text{def}}{=} \det([T]_\beta - t I_n) \in P_n(F)$)

Prop: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T. Then, $\vec{v} \in V$ is an eigenvector of T corresponding to λ iff:

$$\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$$

Pf: Exercise.

$$T\vec{v} = \lambda\vec{v}$$

$$\Leftrightarrow (T - \lambda I_V)\vec{v} = \vec{0}$$

Def: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T.

Then: the subspace $E_\lambda :\stackrel{def}{=} N(T - \lambda I_V) = \{\vec{x} \in V : T(\vec{x}) = \lambda\vec{x}\} \subset V$

is called the <u>eigenspace</u> of T corresponding to λ.

Eigenspaces of a matrix $A \in M_{n\times n}(F)$ is defined as those of L_A

Prop: Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors of T corresponding to $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively, then: $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent.

Proof: We prove by induction on k.

For $k=1$, $\vec{v}_1 \neq \vec{0} \Rightarrow \{\vec{v}_1\}$ is lin. independent.

Suppose the statement holds for $k \geq 1$ distinct eigenvalues.

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}$ be eigenvectors corresponding to $k+1$ distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1}$ of T.

If $\quad a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} = \vec{0}\quad$ for $a_i \in F$,

then applying $T - \lambda_{k+1} I_V$ to both sides $\hat{N}(T-\lambda_{k+1}I_V)\setminus\{\vec{0}\}$ gives:

$$a_1(\lambda_1 - \lambda_{k+1})\vec{v}_1 + \dots + a_k(\lambda_k - \lambda_{k+1})\vec{v}_k = \vec{0}$$

By induction hypothesis,

$$a_1\overset{\neq 0}{(\lambda_1 - \lambda_{k+1})} = \dots = a_k\overset{\neq 0}{(\lambda_k - \lambda_{k+1})} = 0$$

$$\Rightarrow \quad a_1 = a_2 = \dots = a_k = 0$$

$$\Rightarrow \quad a_{k+1}\overset{\neq \vec{0}}{\vec{v}_{k+1}} = \vec{0}$$

$$\Rightarrow \quad a_{k+1} = 0$$

$\therefore\ \{\vec{v}_1, \dots, \vec{v}_{k+1}\}$ is lin. indep.

Corollary: A linear operator T on an n-dim vector space V which has n distinct eigenvalues is diagonalizable.

Proof: Let $\vec{v}_1, \ldots, \vec{v}_n \in V$ be the eigenvectors corresponding to n distinct eigenvalues. Then, the prop. says $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is lin. independent. $\therefore$ $\{\vec{v}_1, \ldots, \vec{v}_n\}$ forms a basis of eigenvectors.

$\therefore$ T is diagonalizable.

Def: Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The **algebraic multiplicity** of λ, denoted $\mu_T(\lambda)$ or $\mu_A(\lambda)$ is the multiplicity of λ as a zero of $f(t)$, i.e. the largest positive integer k s.t. $(t-\lambda)^k \mid f(t)$.

Example: $\cdot$ 1 is eigenvalue of $I_V : \underset{\beta}{V} \to \underset{\beta}{V}$

with $\mu_{I_V}(1) = \dim(V)$

$$f(t) = \det\Big(\underset{\underset{I_n}{\parallel}}{[I_V]_\beta} - t I_n \Big) = \begin{pmatrix} 1-t & & & \\ & 1-t & & \\ & & \ddots & \\ & & & 1-t \end{pmatrix} = (1-t)^n$$

$\cdot$ $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$. $f_A(t) = (3-t)^2 (5-t)$

$\mu_A(3) = 2$, $\mu_A(5) = 1$

Prop: Let T be a linear operator on a finite-dim vector space V and let λ be an eigenvalue of T with algebraic multiplicity $\mu_T(\lambda)$. Then:

$$1 \leq \dim(E_\lambda) \leq \mu_T(\lambda)$$

We call $\gamma_T(\lambda) \overset{\text{def}}{=} \dim(E_\lambda)$ the geometric multiplicity of λ.

Proof: Choose an ordered basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ for E_λ and $\leftarrow p = \dim(E_\lambda)$

extend it to an ordered basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_p, \vec{v}_{p+1}, \ldots, \vec{v}_n\}$ for V.

Then: $[T]_\beta = \begin{pmatrix} | & & | & \\ [\overset{=\lambda\vec{v}_1}{T(\vec{v}_1)}]_\beta & \cdots & [\overset{=\lambda\vec{v}_p}{T(\vec{v}_p)}]_\beta & \cdots \\ | & & | & \end{pmatrix} = \begin{pmatrix} \lambda & 0 & & 0 & \\ 0 & \lambda & & 0 & \\ 0 & 0 & \cdots & \vdots & \\ \vdots & \vdots & & \lambda & \\ 0 & 0 & & 0 & \\ & & & \vdots & \\ & & & 0 & \end{pmatrix}$

$$= \left(\begin{array}{c|c} \lambda I_p & B \\ \hline O & C \end{array}\right)$$

$$\Rightarrow \quad f_T(t) = \det\left(\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline 0 & C - t I_{n-p} \end{array}\right)$$

$$= \det\left((\lambda - t) I_p\right) \underline{\det\left(C - t I_{n-p}\right)}$$

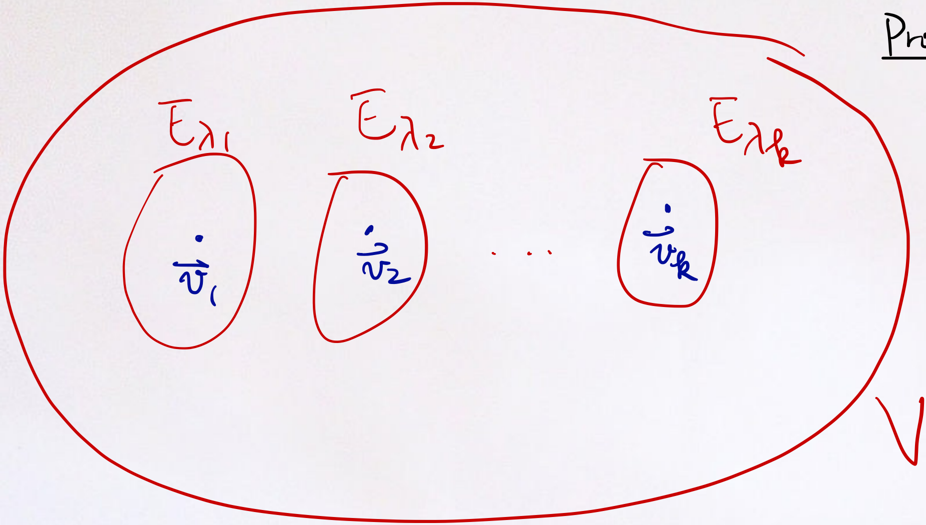
$$= (\lambda - t)^p \det\left(C - t I_{n-p}\right)$$

$$\therefore \quad (\lambda - t)^p \mid f_T(t)$$

$$\therefore \quad \mu_T(\lambda) \geq p = \gamma_T(\lambda)$$

Lemma: Let T be a linear operator, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ distinct eigenvalues of T. For each $i = 1, 2, \dots, k$, let $\vec{v}_i \in E_{\lambda_i}$.

If $\vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_k = \vec{0}$, then $\vec{v}_i = \vec{0}$ for all i.



Proof: If not, say $\vec{v}_1, \dots, \vec{v}_s \neq \vec{0}$

then:

$$\underset{\neq \vec{0}}{\vec{v}_1} + \underset{\neq \vec{0}}{\vec{v}_2} + \dots + \underset{\neq \vec{0}}{\vec{v}_s} = \vec{0}$$

It contradicts to our previous proposition that $\vec{v}_1, \dots, \vec{v}_s$ must be lin. independent.

Proposition: Let T be a linear operator, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, 2, \dots, k$, let $S_i \subset E_{\lambda_i}$ be a finite linearly independent subset. Then:

$S = S_1 \cup S_2 \cup \dots \cup S_k$ is a linearly independent subset of V.

Proof: Write $S_i = \{\vec{v}_{i1}, \vec{v}_{i2}, \dots, \vec{v}_{in_i}\}$ for $i = 1, 2, \dots, k$.

Suppose $\exists\, a_{ij} \in F$ for $1 \le j \le n_i$ and $1 \le i \le k$ such that

$$\sum_{i=1}^{k} \underbrace{\sum_{j=1}^{n_i} a_{ij}\vec{v}_{ij}}_{\overset{\text{def}}{=}\ w_i \,\in\, E_{\lambda_i}} = \vec{0}$$

Then: $\overset{E_{\lambda_1}}{\overset{\in}{w_1}} + \overset{E_{\lambda_2}}{\overset{\in}{w_2}} + \dots + \underset{\in E_{\lambda_k}}{w_k} = \vec{0} \implies w_i = \sum_{j=1}^{n_i} a_{ij}\vec{v}_{ij} = \vec{0}$ for all i.

Then: $a_{ij} = 0$ for all i and j

(for S_i are lin. independent for all i.

$\therefore$ $S_1 \cup S_2 \cup \dots \cup S_k$ is linearly independent.

Theorem: Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial $f_T(t)$ splits.
Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T.

Then: (a) T is diagonalizable iff : $\mu_T(\lambda_i) = \gamma_T(\lambda_i)$
for $i = 1, 2, \ldots, k$

(b) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then $\beta := \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors.

(so that $[T]_\beta$ is a diagonal matrix)

Proof: Write $n = \dim(V)$, and $m_i = \mu_T(\lambda_i)$ and $d_i = \gamma_T(\lambda_i) = \dim(E_{\lambda_i})$ for all i.

Suppose T is diagonalizable and β is a basis for V consisting of eigenvectors of T.

(e.g. $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \dots, \vec{v}_n\}$)

For each i, let $\beta_i = \beta \cap E_{\lambda_i}$ and $n_i \overset{\text{def}}{=} \#\beta_i$

Then: $n_i \le d_i = \dim(E_{\lambda_i})$ ($\because$ β_i is lin. independent)

Also, $d_i \le m_i$

So, we have $n_i \le d_i \le m_i$ for all i.

$$\therefore \quad n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n = \dim(V)$$

$$\therefore \quad \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} n_i = 0 \iff \sum_{i=1}^{k} \overset{1,0}{(d_i - n_i)} = 0$$

$$\Rightarrow \quad d_i = n_i \quad \text{for all } i.$$

$$\therefore \quad \sum_{i=1}^{k} m_i - \sum_{i=1}^{k} d_i = 0 \iff \sum_{i=1}^{k} \overset{1,0}{(m_i - d_i)} = 0$$

$$\Rightarrow \quad d_i = m_i \quad \text{for all } i.$$

$$\therefore \quad n_i = \overset{\dim(E_{\lambda_i})}{\overset{\shortparallel}{d_i}} = m_i \quad \text{for all } i$$

(So, β_i is a basis of E_{λ_i})