Lecture 11:

Recall:

Dual Space Let V be a vector space over F.

- $V^* = \mathcal{L}(V, F)$.
- Let $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for V.

  Then: $\beta^* = \{f_1, f_2, \dots, f_n\}$ is a basis for V^* where

$$f_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- $\dim(V) = \dim(V^*)$
- $\ell : V \to V^{**}$ defined by $\ell(\vec{v})(f) = f(\vec{v})$ where $\vec{v} \in V$ and $f \in V^*$ is an isomorphism.

Definition: Let $T : V \to W$ be linear. The dual map (or transpose) of T is the map $T^* : W^* \to V^*$ defined by:

$$T^*(g) = g(T) \quad \text{for all } g \in W^*.$$

Proposition: Suppose V is fin-dimensional. Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. Let $\beta^* = \{f_1, \ldots, f_n\}$ be the dual basis of β.

Let $T : V \to W$ and γ be the basis of W. Denote the dual basis of γ by γ^*. Then: (1) T^* is linear

(2) $$\underbrace{[T^*]}_{\text{Transpose of } T}{}_{\gamma^*}^{\beta^*} = \underbrace{\left([T]_\beta^\gamma\right)^T}_{\text{Matrix transpose}}$$

$$\begin{tikzcd}
V \arrow[r, "T"] & W \\[-20pt]
\beta & \gamma \\
V^* & W^* \arrow[l, "T^*"'] \\[-20pt]
\beta^* & \gamma^*
\end{tikzcd}$$

Proof: For any $g \in W^*$, $T^*(g) = \underbrace{g}_{\text{linear}} \circ \underbrace{T}_{\text{linear}}$ is linear.

$\therefore$ $T^*(g)$ is a linear functional on V. $\therefore$ $T^*(g) \in V^*$.

Thus: T^* maps W^* to V^*.

T^* is linear: $$T^*(\alpha g_1 + g_2) = (\alpha g_1 + g_2) \circ T$$
$$= \alpha g_1 \circ T + g_2 \circ T = \alpha T^*(g_1) + T^*(g_2)$$

Now, write
$$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$
$$\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$$
$$\beta^* = \{f_1, f_2, \ldots, f_n\}$$
$$\gamma^* = \{g_1, g_2, \ldots, g_n\}$$

Let $A = [T]_\beta^\gamma = (A_{ij})$

To find the j^{th} col of $[T^*]_{\gamma^*}^{\beta^*}$, we write:

$T^*(g_j)$ as a lin. combination of $f_1, f_2, \ldots, f_n$.

Now, $T^*(g_j) = g_j \circ T = \sum_{i=1} (g_j \circ T)(\vec{v}_i) f_i$

$\therefore$ the i^{th}-row, j^{th} col entry of $[T^*]_{\gamma^*}^{\beta^*}$ is given by:

$$g_j \circ T(\vec{v}_i) = g_j\left(\sum_{k=1}^{m} A_{ki} \vec{w}_k\right)$$

$$= \sum_{k=1}^{m} A_{ki}\, g_j(\vec{w}_k) = A_{ji}$$

$\therefore$ $[T^*]_{\gamma^*}^{\beta^*} = A^T = [T]_\beta^\gamma$.

Eigenvalue & Eigenvectors

Def: A linear operator $T: V \to V$ (where V is finite-dim) is called diagonalizable if $\exists$ an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.
A square matrix A is called diagonalizable if L_A is so.

Observation: Say $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

If $\underset{(D_{ij})}{\overset{\parallel}{D}} = [T]_\beta$ is diagonal, then $\forall\, \vec{v}_j \in \beta$, we have:

$$T(\vec{v}_j) = \sum_{i=1}^{n} D_{ij}\vec{v}_i = \underbrace{D_{jj}}_{\lambda_j}\vec{v}_j = \lambda_j \vec{v}_j$$

Conversely, if $T(\vec{v}_j) = \lambda_j \vec{v}_j$ for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$,

then: $[T]_\beta = \begin{pmatrix} | & & \\ [T(\vec{v}_1)]_\beta & - & - \\ | & & \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & \lambda_n \end{pmatrix}$

Def: Let T be a linear operator on a vector space V/F.
A non-zero vector $\vec{v} \in V$ is called an **eigenvector** of T
if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \vec{v}$. In this case, $\lambda \in F$
is called an **eigenvalue** corresponding to the eigenvector $\vec{v}$.

For a square matrix $A \in M_{n \times n}(F)$, a non-zero vector $\vec{v} \in F^n$
is called an eigenvector of A if it is an eigenvector of L_A.

That is: $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in F$.

λ is called the eigenvalue corresponding to the eigenvector $\vec{v}$.

Prop: A linear operator $T: V \to V$ (V = fin-dim) is diagonalizable iff $\exists$ an ordered basis β for V consisting of eigenvectors of T.

In such case, if $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, then:

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

where λ_j is the eigenvalue of T corresponding to $\vec{v}_j$.

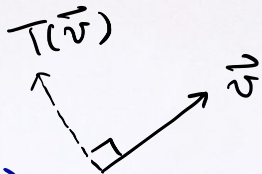
Example:

- $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\beta = \underbrace{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}}_{\text{Check that they are All eigenvectors and } \beta \text{ is basis.}}$

Then: $[L_A]_\beta = \begin{pmatrix} 4 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\frac{\pi}{2}$ in counter-clockwise direction.

(Check: $T = L_A$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)



Then: $T(\vec{v})$ is always perpendicular to $\vec{v}$.

$\therefore$ For $\forall \vec{v} \neq \vec{0}$, it cannot be an eigenvector because:

$T(\vec{v}) \neq \lambda \vec{v}$ for some $\lambda \in F$

Example: Consider $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by:

Space of smooth function

are infinitely differentiable

$$T(f) = f'$$

Then an eigenvector of T with eigenvalue λ is a non-zero solution of:

$$\frac{df}{dt} = \lambda f(t)$$

$$\iff f(t) = C e^{\lambda t} \text{ for some constant } C.$$

$\therefore$ all $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Prop: Let $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Pf: $\lambda \in F$ is an eigenvalue of A

$\iff \exists \vec{v} \in F^n \setminus \{\vec{0}\}$ s.t. $A\vec{v} = \lambda\vec{v}$.

$\iff (A - \lambda I_n)\vec{v} = \vec{0}$

$\iff A - \lambda I_n$ is singular

$\iff A - \lambda I_n$ is not invertible

$\iff \det(A - \lambda I_n) = 0.$

Def: The characteristic polynomial of $A \in M_{n\times n}(F)$ is defined as the polynomial $f_A(t) \overset{def}{=} \det(A - tI_n) \in P_n(F)$

Def: Let T be a linear operator on an n-dim vector space V. Choose an ordered basis β for V. Then, the characteristic polynomial of T is defined as the characteristic polynomial of $[T]_\beta$.

(i.e. $f_T(t) \overset{def}{=} \det([T]_\beta - tI_n) \in P_n(F)$)

Prop: $f_T(t)$ is well-defined, i.e. independent of the choice of β.

Pf: If β' is another ordered basis for V, then:

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \quad \left(Q = [I_V]_{\beta'}^{\beta} \right)$$

Then:

$$\begin{aligned}
\det([T]_{\beta'} - tI_n) &= \det(Q^{-1}[T]_\beta Q - tI_n) \\
&= \det(Q^{-1}([T]_\beta - tI_n)Q) \\
&= \underbrace{\cancel{\det(Q^{-1})}}_{= \frac{1}{\det(Q)}} \det([T]_\beta - tI_n) \cancel{\det(Q)} \\
&= f_T(t).
\end{aligned}$$

Prop: Let $A \in M_{n\times n}(F)$. Then:

- $f_A(t)$ is of degree n and with leading coefficient $(-1)^n$.
- A has at most n distinct eigenvalues.

Pf: Exercise

Def: A polynomial $f(t) \in P(F)$ splits over F if $\exists$ $c, a_1, a_2, \ldots, a_n \in F$ s.t. $f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n)$

Prop: The characteristic polynomial of a diagonalizable linear operator on a finite-dim vector space V/F splits over F.

Pf: If V is a n-dim and $T: V \to V$ is diagonalizable,

then $\exists$ a basis $\beta \subset V$ s.t.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Then: $f_T(t) = \det([T]_\beta - t I_n)$

$$= (-1)^n (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$$

Remark:

- Let T be a linear operator on a vector space V and let λ be an eigenvalue of T. Then, $\vec{v} \in V$ is an eigenvector of T corresponding to λ iff:

$$\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$$

- Next time: We'll discuss <u>eigenspace</u> of T corresponding to λ:

$$E_\lambda \stackrel{\text{def}}{:=} N(T - \lambda I_V) = \{\vec{x} \in V : T(\vec{x}) = \lambda \vec{x}\} \subset V$$