Lecture 10:

Dual Space

Let V be a vector space over F.

Definition: A linear functional on V is a linear map $f: V \to F$.

Remark: A linear functional belongs to $\mathcal{L}(V, F)$.

Definition: The dual space, denoted by V^*, is the space of all linear functional on V. That is, $V^* = \mathcal{L}(V, F)$.

Proposition: Suppose V is finite-dimensional. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. For each $i = 1, 2, \ldots, n$, define a linear functional f_i by setting:

$$f_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then: $\{f_1, f_2, \ldots, f_n\}$ is a basis of V^*, called the dual basis of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. $\therefore \dim(V) = \dim(V^*)$

<u>Proof:</u> • $\{f_1, f_2, \ldots, f_n\}$ is linearly independent.

Suppose : $a_1 f_1 + a_2 f_2 + \ldots + a_n f_n = 0 \leftarrow$ zero functional

For each $\vec{v}_i$,

$$(a_1 f_1 + \ldots + a_n f_n)(\vec{v}_i) = 0 \Rightarrow a_1 f_1(\vec{v}_i) + \ldots + a_n f_n(\vec{v}_i) = 0$$

$$\Rightarrow a_i = 0.$$

$\therefore \{f_1, f_2, \ldots, f_n\}$ is linearly independent.

• $\operatorname{Span}(\{f_1, f_2, \ldots, f_n\}) = V^*$.

Let $f \in V^*$. Suppose $f(\vec{v}_i) = b_i$.

Claim: $b_1 f_1 + b_2 f_2 + \ldots + b_n f_n = f$.

Check : For each $\vec{v}_i$,

$$(b_1 f_1 + \ldots + b_n f_n)(\vec{v}_i) = b_i \cancelto{1}{f_i(\vec{v}_i)} = f(\vec{v}_i) \Rightarrow b_1 f_1 + \ldots + b_n f_n = f.$$

<u>Example:</u> Let $\beta = \{1+x, 1-x, x^2\}$ be the ordered basis for $P_2(\mathbb{R})$

Let $\beta^* = \{f_1, f_2, f_3\}$ be the dual basis of β.

Then:

$$1 = f_1(1+x) = f_1(1) + f_1(x)$$

$$0 = f_1(1-x) = f_1(1) - f_1(x)$$

$$0 = f_1(x^2)$$

Solving: we get $f_1(1) = \frac{1}{2}$, $f_1(x) = \frac{1}{2}$, $f_1(x^2) = 0$

Thus, $f_1(a+bx+cx^2) = a f_1(1) + b f_1(x) + c f_1(x^2)$

$$= \frac{1}{2}a + \frac{1}{2}b$$

f_2 and f_3 can be computed similarly

Remark:

- $\dim(V) = \dim(V^*)$ $\therefore$ V is isomorphic to V^*
  $\uparrow$ fin-dim

- $V^{**} = (V^*)^* =$ dual of the dual space

Proposition: Suppose V is fin-dim. The map $\ell : V \to V^{**}$ defined by $\ell(\vec{v})(f) \overset{\text{def}}{=} f(\vec{v})$ is an isomorphism.
$\in V^*$

Proof: ℓ is linear because: let $\vec{v}_1, \vec{v}_2 \in V$ and $a \in F$. For all $f \in V^*$,

$$\ell(a\vec{v}_1 + \vec{v}_2)(f) = f(a\vec{v}_1 + \vec{v}_2) = a f(\vec{v}_1) + f(\vec{v}_2) \quad (\because f \text{ is linear})$$
$$= a\,\ell(\vec{v}_1)(f) + \ell(\vec{v}_2)(f) = (a\,\ell(\vec{v}_1) + \ell(\vec{v}_2))(f)$$

$$\therefore \ell(a\vec{v}_1 + \vec{v}_2) = a\,\ell(\vec{v}_1) + \ell(\vec{v}_2).$$

To prove that ℓ is an isomorphism, we can just show that ℓ is 1-1 (since $\dim(V) = \dim(V^{**})$)

Suppose $\ell(\vec{v}) = 0$ in V^{**}.

$\Rightarrow \ell(\vec{v})(f) = 0$ for all $f \in V^*$

Then: $f(\vec{v}) = 0$ for all $f \in V^*$

The only possibility is $\vec{v} = \vec{0}$.

(if $\vec{v} \neq \vec{0}$, then construct a basis $\{\vec{v}, \vec{v}_2, \ldots, \vec{v}_n\}$.)
Define $f \to f(\vec{v}) = 1$ and $f(\vec{v}_j) = 0$ for $j = 2, \ldots, n$)

$\therefore$ $\mathrm{Null}(\ell) = \{\vec{0}\}$. Thus, ℓ is 1-1 and onto.

(isomorphism)