- Definition of vector spaces: Try to think about some examples of vector spaces
- Definition of subspaces: Try to think about some examples of subspaces, why is it important?
- What is the linear combination? What is a Spanning set? What is linearly independent? What is the intuitive meaning of linearly dependence? How to check linearly independence?
- What is the definition of basis? What is the meaning of dimension?
- What is the Replacement Theorem? What is the geometric picture of the theorem?
- Try to recall how we can compute RREF? How to compute inverse? How to find the solution set of a linear system? How to determine the dimension of the solution set? What is null-space? What is column space?

Lecture 1: Vector spaces

Field

Definition: A field is a set F along with two binary operations: $+$ (addition) and $\cdot$ (multiplication) such that:

- For $\forall\, x, y \in F$, $x + y = y + x$ and $x \cdot y = y \cdot x$
- For $\forall\, x, y, z \in F$, $(x+y)+z = x+(y+z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- For $\forall\, x, y, z \in F$, $x \cdot (y+z) = x \cdot y + x \cdot z$
- $\exists !$ element $0 \in F \ni \forall\, x \in F$, $x + 0 = x$
- $\exists !$ element $1 \in F \ni \forall\, x \in F$, $x \cdot 1 = x$
- For $\forall\, x \in F$, $\exists$ an element $(-x) \in F \ni x + (-x) = 0$
- For $\forall\, x \in F$ (excluding $x = 0$), $\exists$ an element $x^{-1} \in F \ni x \cdot x^{-1} = 1$

Remark:
- We often write xy for $x \cdot y$
- If F is finite, we say it is a finite field

Examples of field

1. $F = \mathbb{R}$
2. $F = \mathbb{C}$

} Most often considered in Math 2048.

3. $F = \{\text{Rational numbers}\} = \{ p/q : p, q \in \mathbb{Z} \}$
4. Finite field of order p (where p is a prime number)

Define $F_p = \{0, 1, 2, \ldots, p-1\}$ and $+ / \cdot$ are defined as:

$+$: for $\forall x, y \in F_p$, $x + y$ are performed modulo p.

That is, $x+y$ is the remainder of $(x+y)/p$

$\cdot$: for $\forall x, y, \in F_p$, $x \cdot y$ is the remainder of $x \cdot y / p$.

$F_2 = \{0, 1\}$ is the binary field (important for information theories)

Vector Space

Goal: Build an abstract space (space of objects) simulating $\mathbb{R}^n$ or $\mathbb{C}^n$ (with addition and multiplication / scaled)

Definition: A vector space over F is a set V equipped w/ two operations :

(addition) $+ : V \times V \to V$, $(\overset{V}{\vec{x}}, \overset{V}{\vec{y}}) \mapsto \vec{x} + \vec{y} \in V$

(Scalar multiplication) $\cdot : F \times V \to V$, $(\underset{F}{a}, \underset{V}{\vec{x}}) \mapsto a\vec{x} \in V$

satisfying 8 properties:

$+$

(VS1) : $\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \forall \vec{x}, \vec{y} \in V$

(VS2) : $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) \quad \forall \vec{x}, \vec{y}, \vec{z} \in V$

(VS3) : $\exists\ \vec{0} \in V$ s.t. $\vec{x} + \vec{0} = \vec{x} \quad \forall \vec{x} \in V$

(VS4) : $\forall \vec{x} \in V, \ \exists\ \vec{y} \in V$ s.t. $\vec{x} + \vec{y} = \vec{0}$ (inverse)

$\cdot$

(VS5) : $\underset{\underset{F}{\Uparrow}}{1}\,\vec{x} = \vec{x} \quad \forall \vec{x} \in V$

(VS6) : $(\underset{\underset{F}{\Uparrow}}{a}\ \underset{\underset{F}{\Uparrow}}{b})\vec{x} = a(b\vec{x}) \quad \forall a, b \in F, \ \forall \vec{x} \in V$

$+/\cdot$

(VS7) : $\underset{\underset{F}{\Uparrow}}{a}\,(\underset{\underset{V}{\Uparrow}}{\vec{x}} + \underset{\underset{V}{\Uparrow}}{\vec{y}}) = a\vec{x} + a\vec{y} \quad \forall a \in F, \ \forall \vec{x}, \vec{y} \in V$

(VS8) : $(a + b)\vec{x} = a\vec{x} + b\vec{x} \quad \forall a, b \in F, \ \forall \vec{x} \in V$

Remark: an element in F is called scalar.
an element in V is called vector.

Examples of vector spaces

- $F^n = \{(x_1, x_2, \dots, x_n) : x_j \in F \text{ for } j = 1, 2, \dots, n\}$ w/

$$(x_1, x_2, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$a(x_1, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$$

- $M_{m\times n}(F) = \{ m \times n$ matrices w/ entries in $F\}$ w/ matrix addition and scalar multiplication

- $P(F) = \{$ polynomials w/ coefficients in $F\}$ w/ polynomial addition and scalar multiplication.

- $F^\infty = \{(x_1, x_2, \dots) : x_j \in F, j = 1, 2, \dots\}$ w/ component-wise addition and scalar multiplication.

- $\mathrm{Sym}_{n\times n}(F) = \{ n\times n$ symmetric matrices A w/ entries in F : $A^T = A\}$

- Let S be any non-empty set.

Then: $\mathcal{F}(S, F) = \{$ functions $f : S \to F\}$

is a vector space over F under:

$$\underset{\mathcal{F}(S,F)}{(f} + \underset{\mathcal{F}(S,F)}{g)}(\underset{}{s}^{\in S}) \overset{\text{def}}{=} f(s) + g(s) \; ; \qquad (\underset{F}{a} f)(s) \overset{\text{def}}{=} a f(s).$$

- $\mathbb{C}$ is a vector space over $F = \mathbb{C}$

__Question__: Is $V = \mathbb{R}$ a vector space over $F = \mathbb{C}$?? No

$2 \in V$, $i \in F$ but $i \cdot 2 \notin V$.

- Consider the differential equation:

$$(*) \quad \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \qquad (a, b \in \mathbb{R})$$

Let S be the set of twice differentiable functions on $\mathbb{R}$ satisfying $(*)$.

Then S is a vector space under usual addition and scalar multiplication is a vector space.

<u>Proposition:</u> Let V be a vector space over F. Then:

(a) The element $\vec{0}$ in (VS3) is unique, called <u>zero vector</u>

(b) $\forall \vec{x} \in V$, the element $\vec{y}$ in (VS4) is unique, called the <u>additive inverse</u> (Denoted as $-\vec{x}$)

(c) $\vec{x} + \vec{z} = \vec{y} + \vec{z} \Rightarrow \vec{x} = \vec{y}$ (Cancellation law)

(d) $\underset{\substack{\uparrow \\ F}}{0}\vec{x} = \vec{0} \quad \forall \vec{x} \in V$

(e) $\underset{\substack{\uparrow \\ F}}{(-a)}\vec{x} = -(a\vec{x}) = a(-\vec{x}), \quad \forall a \in F, \quad \forall \vec{x} \in V$

(f) $\underset{\substack{\uparrow \\ F}}{a}\vec{0} = \vec{0} \quad \forall a \in F$

Subspace

Definition: A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F under the same addition and scalar multiplication inherited from V.

Proposition: Let V be a vector space V over F. A subset $W \subset V$ is a subspace iff the following 3 conditions hold:

(a) $\vec{0}_V \in W$

(b) $\vec{x} + \vec{y} \in W$, $\forall \vec{x}, \vec{y} \in W$ (closed under $+$)

(c) $a\vec{x} \in W$, $\forall a \in F$, $\forall \vec{x} \in W$ (closed under $\cdot$)

Examples:

- For any vector space V / F,

  $\{\vec{0}\} \subset V$; $V \subset V$ (trivial subspaces)

  "zero subspace"

- For $V = M_{n \times n}(F)$,

  $W_1 = \{\text{diagonal matrices}\} \subset V$ subspace

  $W_2 = \{A \in M_{n \times n}(F) : \det(A) = 0\} \subset V$ is not a subspace.

  $W_3 = \{A \in M_{n \times n}(F) : \operatorname{tr}(A) = 0\} \subset V$

  subspace

- For $V = P(F)$

$P_n(F) \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) \leq n \}$ is a subspace

$W \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) = n \}$ is NOT a subspace

- Consider $V = F^n = \{ \underbrace{(x_1, x_2, \ldots, x_n)}_{= \vec{x}^T} : x_j \in F \text{ for } j = 1, 2, \ldots, n \}$

Consider linear system:

$$\left\{ \begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + \cdots\cdots\cdots + a_{2n} x_n &= b_2 \\ &\;\;\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m \end{aligned} \right. \iff A\vec{x} = \vec{b}$$

gives a subset, the solution set $S \subset V$

Is S a subspace? No in general.

Yes only when $\vec{b} = \vec{0}$.

Theorem: Any intersection of subspaces of a vector space V is a subspace of V.

Question: W_1 = subspace ; W_2 = subspace

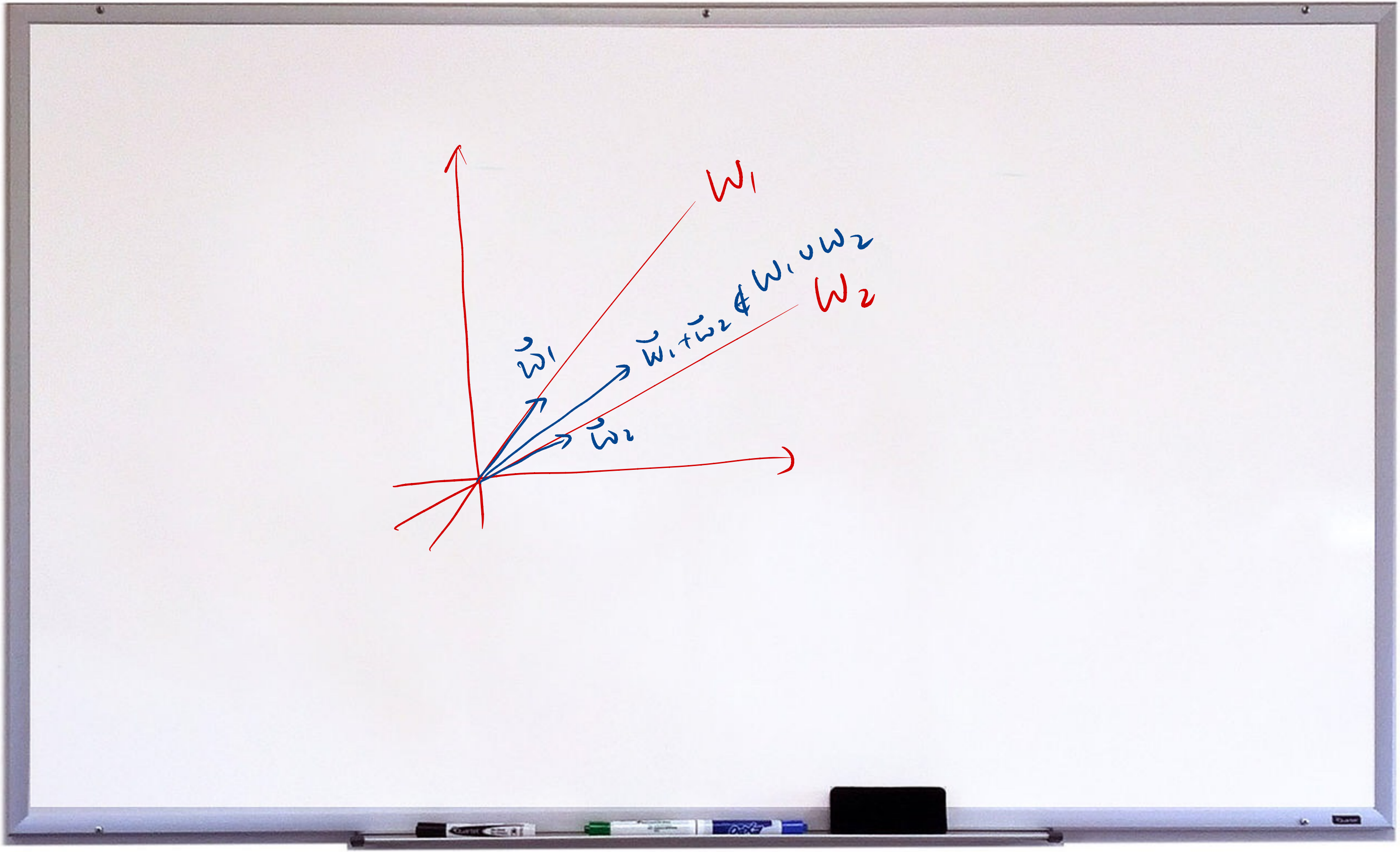
$W_1 \subseteq V$; $W_2 \subseteq V$

$\Downarrow$

$W_1 \cap W_2$ is subspace

Is $W_1 \cup W_2$ a subspace ?? No in general

$W_1 \cup W_2 \subseteq V$


W_1
$\vec{w}_1 + \vec{w}_2 \notin W_1 \cup W_2$
W_2
$\vec{w}_1$
$\vec{w}_2$

Linear combination and Span

Definition: Let V be a vector space over F and $S \subset V$ a non-empty subset.

- We say a vector $\vec{v} \in V$ is a <u>linear combination</u> of vectors of S if $\exists\ \vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$ and $a_1, a_2, \dots, a_n \in F$ such that:

$$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n.$$

Remark: $\vec{v}$ is usually called a linear combination of $\vec{u}_1, \dots, \vec{u}_n$ and $a_1, \dots, a_n$ are the coefficients of the linear combination.

- The <u>span</u> of S, denoted as $\text{Span}(S)$, is the set of all linear combination of vectors of S.

$$\text{Span}(S) \overset{\text{def}}{=} \{ a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n : a_j \in F,\ \vec{u}_j \in S \text{ for } j = 1, 2, \dots, n,\ n \in \mathbb{N} \}$$

Remark:

- By convention, $\text{span}(\phi) \overset{\text{def}}{=} \{\vec{0}\}$.

  ϕ = empty set

e.g. $1 \in \text{Span}(\{1+x^2, 1-x^2\})$

x

Example:

- $F^n = \operatorname{Span}(\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\})$ where $\vec{e}_j = (0, 0, \cdots, \underset{\uparrow \atop j^{th}}{1}, 0, \ldots 0)$

- $P(F) = \operatorname{Span}(\{1, x, x^2, \ldots, x^n, \ldots\})$

- $M_{n\times n}(F) = \operatorname{Span}(S)$

$$S = \left\{ E_{ij} \overset{\text{def}}{=} \begin{array}{cc} \begin{matrix} \quad & \quad & \quad & \downarrow j^{th} & \quad \end{matrix} & \\ \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & & 0 \\ 0 & \cdots & \cdots & 1 & 0 \\ & & & & \\ 0 & \cdots & \cdots & & 0 \end{pmatrix} & \leftarrow i^{th} \end{array} : 1 \le i, j \le n \right\}$$

- Given $\vec{u}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \vec{u}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$

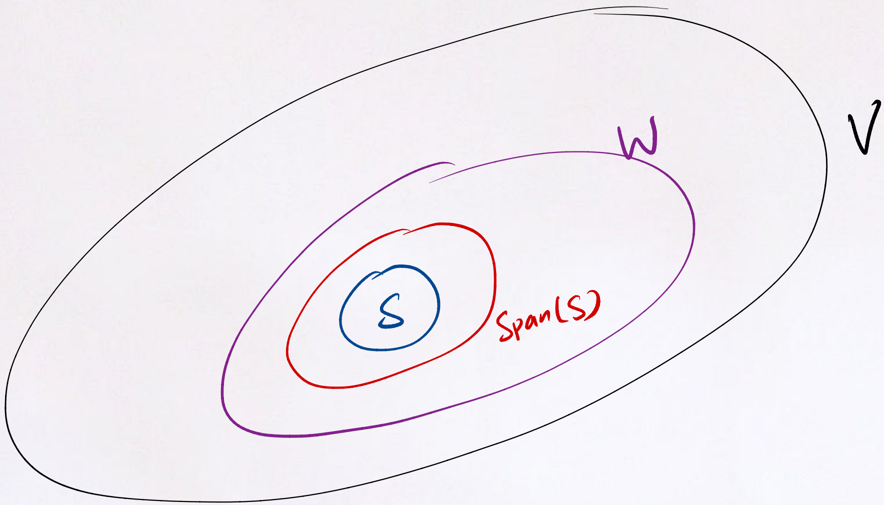
Then: $\vec{v} \in \operatorname{Span}(\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\})$ iff

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = v_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = v_2 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = v_n \end{cases}$$

has a sol.

Theorem: Let $S \subset V$ be a subset of a vector space V over F.
Then, span(S) is the smallest[1] subspace[2] of V consisting S.
(If W is a subspace containing S, then Span$(S) \subset W$)



Linear independence

Definition: Let V be a vector space over F. A subset $S \subset V$ is said to be linearly dependent if $\exists$ distinct $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$, not all zero, s.t.

$$a_1 \vec{u}_1 + a_2 \vec{u}_2 + \ldots + a_n \vec{u}_n = \vec{0}$$

Otherwise, it is said to be linearly independent.

e.g.

- The empty set $\phi \subset V$ is linearly
- If $\vec{0} \in S$, the S is linearly
- If $S = \{\vec{u}\}$ and $\vec{u} \neq \vec{0}$, then S is linearly independent.

Proposition: Let $S \subset V$ be a subset of a vector space V. Then, the following are equivalent.

(1) S is linearly independent

(2) Each $\vec{x} \in \text{span}(S)$ can be expressed in a unique way as a linear combination of vectors of S.

(3) The only representations of $\vec{0}$ as linear combinations of vectors of S are trivial representations, i.e., if

$$\vec{0} = a_1 \vec{u}_1 + \cdots + a_n \vec{u}_n \text{ for}$$

some $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$, $a_1, a_2, \ldots, a_n \in F$, then we must have $a_1 = a_2 = \cdots = a_n = 0$

Example: • For $k = 0, 1, 2, \ldots, n$, let $f_k(x) = 1 + x + x^2 + \cdots + x^k$.

Then: $S = \{ f_0(x), f_1(x), f_2(x), \ldots, f_n(x) \} \subset P_n(F)$ is a linearly independent subset.

Exercise.

Theorem: Let S be a linearly independent subset of a vector space V. Let $\vec{v} \in V \setminus S$. Then: $S \cup \{\vec{v}\}$ is linearly dependent iff $\vec{v} \in \operatorname{Span}(S)$.

Definition: A **basis** for a vector space V is a subset $\beta \subset V$ such that:

- β is linearly independent and
- β spans V, i.e. $\mathrm{Span}(\beta) = V$.

e.g.
- F^n: $\{\vec{e}_1 = (1, 0, \ldots, 0), \vec{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \vec{e}_i = (0, \ldots, 0, \underset{\text{i-th}}{1}, 0 \ldots 0), \ldots, \vec{e}_n = (0, 0, \ldots, 1)\}$ is a basis for F^n. (Standard basis)
- $M_{2\times 2}(F)$: $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \subset M_{2\times 2}(F)$ is a basis for $M_{2\times 2}(F)$
- $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(F)$
- $\{1, x, x^2, \ldots\}$ is a basis for $P(F)$.

Theorem: Let V be a vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \subset V$.

Then: β is basis for V if and only if : $\forall \vec{v} \in V$, $\exists!$ (unique)

(for all) (in) (there exist)

$a_1, a_2, \ldots, a_n \in F$ such that :

$$\vec{v} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \ldots + a_n \vec{u}_n.$$

Lemma: Let S be a linearly dependent subset of a vector space V.
Then: $\exists\, \vec{v} \in S$ such that $\operatorname{span}(S \setminus \{\vec{v}\}) = \operatorname{span}(S)$.

Theorem: Suppose S is a finite spanning set for a vector space V.
Then: $\exists\, \beta \subset S$ which is a basis for V.
(A finite spanning set can be reduced to a basis)