MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 6

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-10-25 (Friday) 23:59****.*

1. Let $T \in \mathcal{L}(P_2(\mathbb{R}))$ be defined by $T(f(x)) = af(0) + f(-1)(x + x^2)$. Prove that T is not diagonalizable for any $a \in \mathbb{R}$.

   Proof. For any real c_1, c_2, and c_3, $T(c_1x^2 + c_2x + c_3) = ac_3 + (c_1 - c_2 + c_3)(x + x^2)$. The characteristic polynomial is then $x^2(a - x)$. By some computation of the dimension of eigenspaces, we see T is not diagonalizable for any a.

2. Let $A \in M_{n\times n}(F)$.

   (a) Show that A and A^T have the same characteristic polynomials and eigenvalues.
   Proof. Notice that $p_A(t) = \det(A - tI) = \det((A - tI)^T) = \det(A^T - tI) = p_{A^T}(t)$.

   (b) Give an example that A and A^T could have different eigenspaces for a given common eigenvalue.
   Proof. $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

   (c) Prove that for any common eigenvalue λ, $\gamma_A(\lambda) = \gamma_{A^T}(\lambda)$.
   Proof. This follows from the rank-nullity theorem and the fact that $A - \lambda I$ and $A^T - \lambda I$ share the same rank.

   (d) Prove that if A is diagonalizable, then A^T is also diagonalizable.
   Proof. By (a), A and A^T have the same eigenvalues. Further, (c) implies that for each eigenvalue λ, the dimension of the corresponding eigenspaces of A and A^T also coincide. The result then follows.

3. Let A be a $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with corresponding multiplicities $m_1, m_2, .., m_k$. Prove the following statements.

   (a) $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$
   Proof. Write $A = P^{-1}UP$, where P is invertible and U is upper triangular with distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with corresponding multiplicities $m_1, m_2, .., m_k$. We will use a basic property of trace. That is, for any square matrices A and B, we have $tr(AB) = tr(BA)$. So, $tr(A) = tr(P^{-1}UP) = tr(UP^{-1}P) = tr(U) = \sum_{i=1}^{k} m_i\lambda_i$.

(b) $\det(A) = \prod_{i=1}^{k} \lambda_i^{m_i}$

Proof. $\det(A) = \det(P^{-1}UP) = \det(P^{-1})\det(U)\det(P) = \det(U) = \prod_{i=1}^{k} \lambda_i^{m_i}$, where we have used $\det(P^{-1}) = \det(P)^{-1}$.

4. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, ..., \lambda_k$.

   (a) Prove that span($\{x \in V : x$ is an eigenvector of $T\} = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots E_{\lambda_k}$.

   Proof. The left hand side is definitely the Minkowski sum of all the eigenspaces. It remains to prove that it is actually a direct sum, which can be guaranteed if $E_{\lambda_1} \cup E_{\lambda_2} = \{0\}$ for any $\lambda_1 \neq \lambda_2$. Suppose $v \in E_{\lambda_1}$ and also $v \in E_{\lambda_2}$, then $\lambda_1 v = T(v) = \lambda_2 v$. So we must have $v = 0$.

   (b) Hence, prove that $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots E_{\lambda_k}$ if T is diagonalizable.

   Proof. If T is diagnalizable, the dimension of the left hand side of (a) coincides with the dimension of V. We immediately get the desired result.

5. Let T be a linear operator on a vector space V, and suppose there exist linearly independent non-zero vectors $u, v \in V$ such that $T(u) = 2v$ and $T(v) = 2u$. Prove that 2 and -2 are eigenvalues of T.
   Hint: Construct eigenvectors corresponding to the eigenvalues.

   Proof. Notice that $T(u + v) = T(u) + T(v) = 2(T(u) + T(v)) = 2T(u + v)$, and similarly $T(u - v) = -2T(u - v)$. So, 2 and -2 are eigenvalues of T. The condition that u and v are linearly independent is used implicitly to ensure $u + v$ and $u - v$ are not zero vectors.