MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 5

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-10-18 (Friday) 23:59****.*

1. Define $f \in (\mathbb{R}^2)^*$ by $f(x,y) = 2x + y$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (3x+2y, x)$.

   (a) Compute $T^*(f)$.

   Proof. Since $T^*(e^1)(x,y) = e^1(T)(x,y) = 3x + 2y$, and $T^*(e^2)(x,y) = e^2(T)(x,y) = x$, $T^*(f)(x,y) = 2(3x+2y) + x = 7x + 4y$.

   (b) Let β be the standard ordered basis for $\mathbb{R}^2$ and $\beta^* = \{f_1, f_2\}$ be the dual basis. Compute $[T^*]_{\beta^*}$ by expressing $T^*(f_1)$ and $T^*(f_2)$ as linear combinations of f_1 and f_2.

   Proof. $T^*(f_1) = 3f_1 + f_2$ and $T^*(f_2) = f_1$.

   (c) What is the relationship between $[T]_\beta$ and $[T^*]_{\beta^*}$?

   Proof. Transpose of each other.

2. Prove that a function $T : F^n \to F^m$ is linear if and only if there exist $f_1, f_2, ..., f_m \in (F^n)^*$ such that $T(x) = (f_1(x), f_2(x), ..., f_m(x))$ for all $x \in F^n$.

   Proof. Let $v_1, \cdots, v_n$ be the standard basis of F^n. The forward direction follows by defining, for each $1 \leq k \leq m$, $f_k(v_i) = (T(v_i))_k$ for $1 \leq i \leq n$. The reverse direction follows the linearity of each f_k.

3. Let V an W be finite-dimensional vector spaces over F. Let $\psi_1 : V \to V^{**}$ be defined by $\psi_1(v)(f) = f(v)$ for all $f \in V^*$ and $\psi_2 : W \to W^{**}$ be defined by $\psi_2(w)(g) = g(w)$ for all $g \in W^*$. Note that ψ_1 and ψ_2 are isomorphisms.
   Let $T : V \to W$ be linear, and define $T^{**} = (T^*)^*$. Prove that $\psi_2 T = T^{**}\psi_1$.

   Proof. For any $v \in V$ and $g \in W^*$, $\psi_2 T(v)(g) = g(T(v))$ and $T^{**}\psi_1(v)(g) = (\psi_1(v)T^*)(g) = T^*g(v) = g(T(v))$. Hence, two operators coincide.

4. Given the matrix

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}.$$

   (a) Find the characteristic polynomial $f_A(x)$, then prove that $f_A(x)$ splits.

   Proof. $f_A(x) = \det(A - xI) = -(x-1)(x-2)(x-3)$

(b) Determine all the eigenvalues of A, then find the set of eigenvectors corresponding to λ for each eigenvalue λ of A.

Proof. The eigenvalues are 1, 2, and 3. The corresponding eigenvectors are $(1, 1, -1)$, $(1, -1, 0)$, and $(1, 0, -1)$ respectively.

(c) Show that there exist a basis for $\mathbb{R}^3$ consisting of eigenvectors of A, then find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Proof. Denote the eigenvectors found in (b) by v_1, v_2, and v_3. It suffices to let $Q = [v_1, v_2, v_3]$. The diagonal entries of D are 1, 2, and 3 from left to right.

5. Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F.

(a) Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(x) = g(\lambda)(x)$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue g(λ).

Proof. Let $g(t) = a_n t^n + \cdots + a_1 t + a_0$. Then, for any eigenvector x corresponding to eigenvalue λ, $g(T)(x) = a_n T^n(x) + \cdots + a_1 T(x) + a_0 = a_n \lambda^n x + \cdots + a_1 \lambda x + a_0 = g(\lambda)(x)$.

(b) Let f_T be the characteristic polynomial of T. Prove that if T is diagonalizable, then $f(T) = T_0$, the zero operator. (We will see that this result does not depend on the diagonalizability of T in later sections.)

Proof. Since T is diagonalizable, we can find an eigenbasis of T, consisting of $v_1, \cdots, v_n$. For each v_k, $k = 1, \cdots, n$, we have $f(T)(v_k) = 0$. So. $f(T)$ is simply the zero operator.