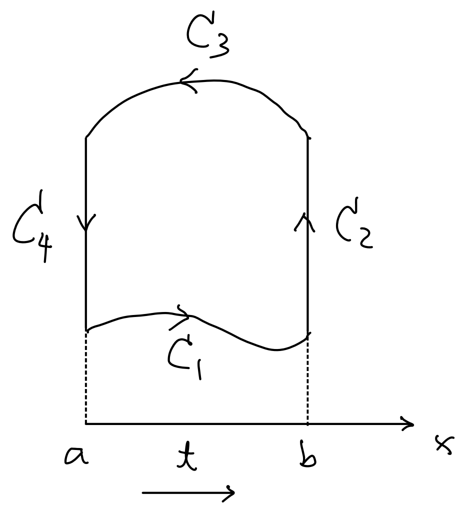
Pf of Green's Thm for Simple Region

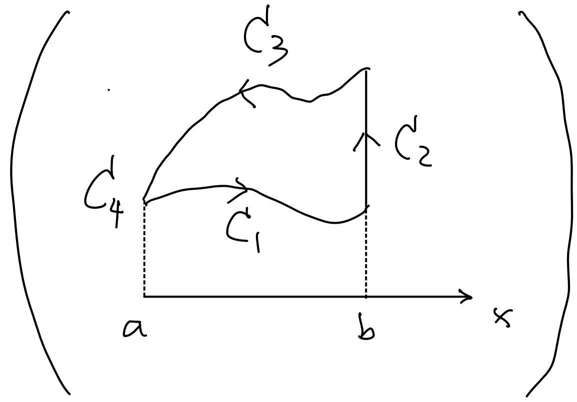


By definition, R is of type (I) and can be written as

$R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Let denote the components of the boundary of R by C_1, C_2, C_3 & C_4 as in the figure

(Note: C_2 and/or C_4 could just be a point)



Then $\partial R = C_1 + C_2 + C_3 + C_4$ as oriented curves

(using "+" instead of "$\cup$" to denote the orientation)

Now $C_1 = \{y = g_1(x)\}$ can be parametrized by

$(x,y): \ \vec{r}(t) = (t, g_1(t))$, $\quad a \le t \le b$

(with correct orientation)

Similarly "$-C_3$" can be parametrized by

$\vec{r}(t) = (t, g_2(t))$, $\quad a \le t \le b$ (with correct orientation)

Hence $\displaystyle\int_{C_1} M dx = \int_a^b M(t, g_1(t))\,dt$

and $\displaystyle\int_{-C_3} M dx = \int_a^b M(t, g_2(t))\,dt$

For $C_2 = \{ x=b,\ g_1(b) \le y \le g_2(b) \}$, it can be parametrized by

$\vec{r}(t) = (b, t)$, $g_1(b) \le t \le g_2(b)$

(with correct orientation)

Similarly for $-C_4 = \{ x=a,\ g_1(a) \le y \le g_2(a) \}$:

$\vec{r}(t) = (a, t)$, $g_1(a) \le t \le g_2(b)$

(with correct orientation)

Hence $\displaystyle\int_{C_2} M dx = \int_{g_1(b)}^{g_2(b)} M(b, t)\,db = 0$

$\displaystyle\int_{-C_4} M dx = \int_{g_1(a)}^{g_2(a)} M(a, t)\,da = 0$

Therefore $\displaystyle\oint_{\partial R} M dx = \sum_{i=1}^{4} \int_{C_i} M dx$

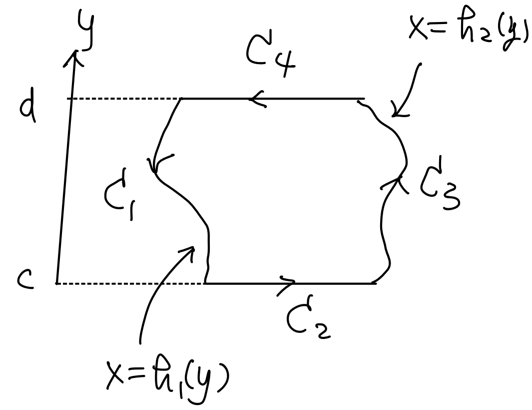
$\displaystyle= \int_a^b [M(t, g_1(t)) - M(t, g_2(t))]\,dt$

On the other hand, Fubini's Thm $\Rightarrow$

$$\iint_R -\frac{\partial M}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \left(-\frac{\partial M}{\partial y}\right) dy\, dx$$

$$= \int_a^b -\Big(M(x, g_2(x)) - M(x, g_1(x))\Big)\, dx$$

$$= \oint_{\partial R} M dx$$



Similarly, R is also type (2)

$R = \{(x,y) : h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$

$$\oint_{\partial R} N dy = -\int_c^d N(h_1(t), t)\, dt + 0$$

$$+ \int_c^d N(h_2(t), t))\, dt + 0$$

$$= \int_c^d \left[N(h_2(t), t) - N(h_1(t), t)\right] dt$$

$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\, dx \right) dy$$

$$= \iint_R \frac{\partial N}{\partial x}\, dA$$

All together

$$\oint_{\partial R} M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

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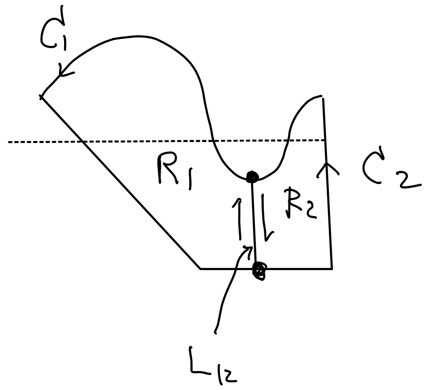
Proof of Green's Thm for

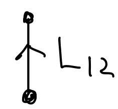
$R =$ finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most.

eg:



R_1, R_2 are simple but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L_{12}$



$\partial R_2 = C_2 - L_{12}$

with anti-clockwise orientation

and $\partial R = C_1 + C_2$

By assumption $R = \bigcup R_i$ finite union s.t.

- R_i are simple, and
- $R_i \cap R_j =$ line segment of a common boundary portion denoted by L_{ij} $(i \neq j)$

Then $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \sum_i \oint_{\partial R_i} M dx + N dy \quad \left(\begin{array}{l}\text{by Green's Thm} \\ \text{for simple region}\end{array}\right)$$

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except possibly at the end points)

Then $\partial R_i = C_i + \sum_{j \neq i} L_{ij}$

where L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i

Hence $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \oint_{C_i + \sum_{j \neq i} L_{ij}} M dx + N dy$$

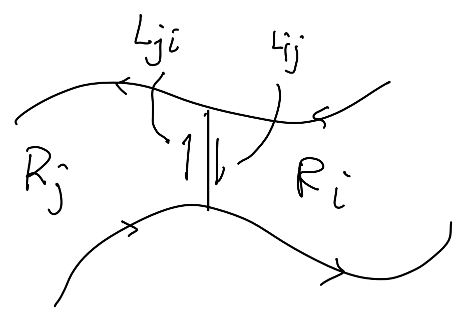
$$= \sum_i \int_{C_i} M dx + N dy + \sum_i \sum_{j \neq i} \int_{L_{ij}} M dx + N dy$$

Note that, as C_i is not a common boundary of any other R_j

$$\sum_i C_i = \partial R$$

$$\therefore \quad \sum_i \int_{C_i} M dx + N dy = \oint_{\partial R} M dx + N dy$$

Finally, we have $L_{ij} = -L_{ji}$ as R_i and R_j are located on the two different sides of the common boundary.



$$\sum_{i} \sum_{j \neq i} \int_{L_{ij}} Mdx + Ndy = \sum_{\substack{i,j \\ i \neq j}} \int_{L_{ij}} Mdx + Ndy$$

$$= \sum_{i<j} \int_{L_{ij}} Mdx + Ndy + \sum_{\substack{\cancel{j<i} \\ i \quad j}} \int_{\substack{L_{\cancel{ij}} \\ ji}} Mdx + Ndy$$

$$= \sum_{i<j} \left(\int_{L_{ij}} Mdx + Ndy + \int_{L_{ji}} Mdx + Ndy \right)$$

$$= \sum_{i<j} \left(\int_{L_{ij}} Mdx + Ndy - \int_{L_{ij}} Mdx + Ndy \right)$$

since $L_{ij} = -L_{ji}$

$$= 0$$

This 2^{nd} case basically include almost all situations in the level of Advanced Calculus.

The proof of <u>general case</u> needs "analysis" and will be omitted here.

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$\boxed{\begin{array}{c}\underline{\text{Def 12}}: \text{The } \underline{\text{divergence}} \text{ of } \vec{F} = M\hat{i} + N\hat{j} \text{ is defined to be} \\ \operatorname{div}\vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\end{array}}$

<u>Note</u>:

$$\operatorname{div}\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\overline{D}_\varepsilon(x,y))} \iint_{\overline{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\overline{D}_\varepsilon(x,y))} \oint_{\partial\overline{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{n}\, ds$$

($\overline{D}_\varepsilon(x,y)$ = closed disk of radius ε centered at (x,y))

called "flux density".

<u>Notation</u>: For $f(x,y)$,

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \quad \text{(gradient)}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f$$

It is convenient to denote

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)$$

Then

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot \left(M\hat{i} + N\hat{j}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}$$

Hence we also write

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F}}$$

Def 13: Define rot $\vec{F}$ to be

$$\text{rot}\,\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad (\text{for } \vec{F} = M\hat{i} + N\hat{j})$$

Note: $\text{rot}\,\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\overline{D_\varepsilon}(x,y))} \iint_{\overline{D_\varepsilon}(x,y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\overline{D_\varepsilon}(x,y))} \oint_{\partial \overline{D_\varepsilon}(x,y)} \vec{F} \cdot \hat{T}\, ds$$

$\overset{\text{called}}{=}$ circulation density

Using $\vec{\nabla}$, we can write

$$\boxed{\text{rot}\,\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}}$$

Since $\vec{F} = M\hat{i} + N\hat{j} + 0\hat{k}$ number zero $\left(\text{in } \mathbb{R}^3,\ \begin{matrix} M = M(x,y) \\ N = N(x,y) \end{matrix}\right)$

$\left(\begin{matrix}\text{generalize} \\ \text{2-dim to 3-dim}\end{matrix}\right)$ $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ $\left(\text{so that } \vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$

$$\Rightarrow \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix} \hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$\Rightarrow$ $\text{rot}\,\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$ (ie. k-component of $\vec{\nabla} \times \vec{F}$)

A name for $\vec{\nabla} \times \vec{F}$ is <u>curl $\vec{F}$</u> : $\boxed{\text{curl}\,\vec{F} \overset{\text{def}}{=} \vec{\nabla} \times \vec{F}}$

In these notation, the Green's thm can be written as

<u>Vector form of Green's Thm</u>

<u>normal form</u>

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA$$

or

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \vec{\nabla} \cdot \vec{F}\, dA$$

<u>tangential form</u>

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_R \operatorname{curl} \vec{F} \cdot \hat{k}\, dA$$

or

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dA$$

And Thm 10 can be written as

<u>Thm 10′</u>: Ω <u>simply-connected</u> & <u>connected</u>, $\vec{F} \in C^1$.

Then $\vec{F}$ = conservative $\Longleftrightarrow$ $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \vec{0}$ (check)

<u>Note</u>: (i) $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$ defined only in $\mathbb{R}^3$ $(\supset \mathbb{R}^2)$

(ii) but $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$ can be defined on $\mathbb{R}^n$ for any n

In particular, in $\mathbb{R}^3$

<u>Def 12′</u> The <u>divergence</u> of $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ is defined to be

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (M\hat{i} + N\hat{j} + L\hat{k}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}$$

Then one can easily check the following facts: (Ex!)

$$\boxed{\begin{array}{l} \text{(i)}\ \vec{\nabla}\times(\vec{\nabla}f)=\vec{0}\quad (\text{i.e. curl}\ \vec{\nabla}f=\vec{0}) \\ \text{(ii)}\ \vec{F}\ \text{conservative} \Rightarrow \text{curl}\,\vec{F}=\vec{\nabla}\times\vec{F}=\vec{0} \\ \text{(iii)}\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})=0\quad (\text{i.e. div}(\text{curl}\,\vec{F})=0) \end{array}}$$

<u>Remark</u>: $\vec{\nabla}\cdot(\vec{\nabla}f)\neq 0$ in general, and it is called the

<u>Laplacian</u> of f and is denoted by

$$\boxed{\begin{aligned} \vec{\nabla}^2 f &= \vec{\nabla}\cdot(\vec{\nabla}f) = \text{div}(\vec{\nabla}f) \\ &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \end{aligned}}$$

[In graduate level, it will be denoted by $\Delta=\vec{\nabla}^2$ or $\Delta=-\vec{\nabla}^2$]

The "operator" $\vec{\nabla}^2$ is called the <u>Laplace operator</u> and the equation $\vec{\nabla}^2 f=0$ is called the <u>Laplace equation</u>. Solutions to the Laplace equation are called <u>harmonic functions</u>.