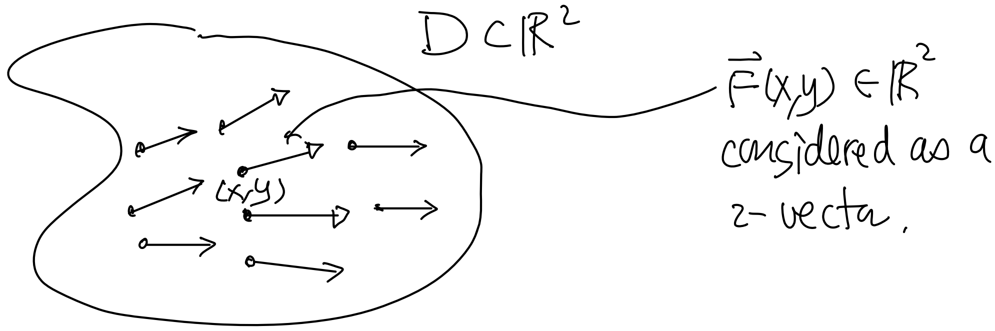
Vector Fields

Def 10: Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a <u>vector field</u> on D is a mapping $\vec{F} : D \to \mathbb{R}^2$ or $\mathbb{R}^3$ respectively



In <u>component form</u>:

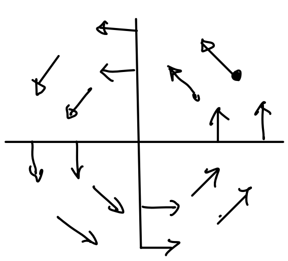
$$\mathbb{R}^2: \quad \vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$$

$$\mathbb{R}^3: \quad \vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + L(x,y,z)\,\hat{k}$$

where M, N, L are functions on D called the <u>components</u> of $\vec{F}$.

<u>eg 35</u> $\quad \vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

$$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad \text{(in polar coordinates)}$$



Properties of $\vec{F}$: (i) <u>$\|\vec{F}(x,y)\| = 1$</u>

(ii) <u>$\vec{F} \perp \vec{r}(x,y)$</u> $= x\hat{i} + y\hat{j} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

eg36 (Gradient vector field of a function)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) \overset{\text{def}}{=\!=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i} + y\hat{j} = \vec{r}(x,y)$$

position vector field.

(ii) $f(x,y,z) = x$

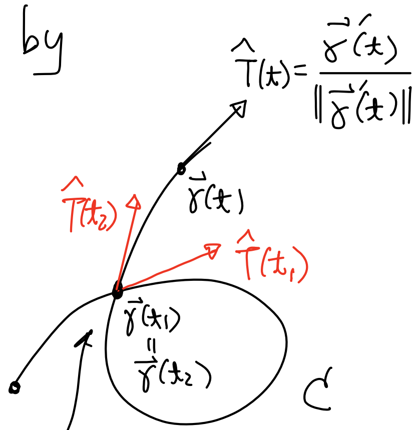
$$\vec{\nabla} f(x,y,z) \overset{\text{def}}{=\!=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \equiv (1,0,0) = \hat{i}$$

(a constant vector field)

eg37 (Vector field along a curve)

Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\begin{aligned} \vec{\gamma} : [a,b] &\to \mathbb{R}^2 \\ t &\mapsto (x(t),y(t)) = \vec{\gamma}(t) \end{aligned}$$



Recall: $\hat{T} = \dfrac{\vec{\gamma}'(t)}{\|\vec{\gamma}'(t)\|}$ (provided $\vec{\gamma}'(t) \neq 0$)

= unit tangent vector field along C (same point, but different vector)

Note: this $\hat{T}$ defined only on C (for a general curve), but not outside C.

(vector field along a curve may not come from a vector field on a region.)

Remark: for eg 37.

If we use $ds = \|\vec{r}'(t)\| dt$, then

$$\hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds} \quad \text{(by Chain rule)}$$

(if s is a function of t)

where "arc-length s" is defined by

(up to an additive constant)

$$s(t) = \int_{t_0}^{t} \|\vec{r}'(t)\| dt,$$

A parametrization of a curve C by arc-length s is called arc-length parametrization:

$$\vec{r}(s) = \text{arc-length parametrization}$$

$$\Rightarrow \left\| \frac{d\vec{r}}{ds}(s) \right\| = 1$$

Def 11 A vector field is defined to be continuous / differentiable / C^k if the component functions are.

eg38:

$$\begin{cases} \vec{F}(x,y) = \vec{r}(x,y) = x\hat{i} + y\hat{j} \text{ is } C^\infty \text{ (position vector)} \\ \vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}} \text{ is not continuous in } \mathbb{R}^2 \\ \quad (\text{but continuous in } \mathbb{R}^2 \setminus \{(0,0)\}) \end{cases}$$

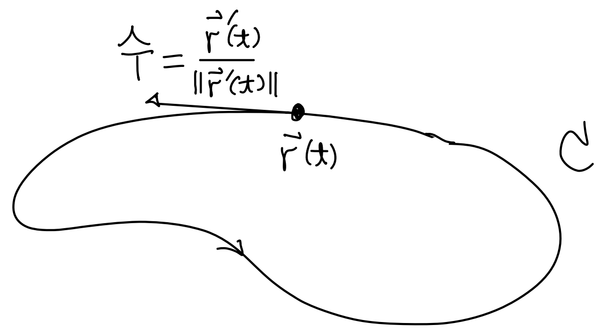
Line integral of vector field

Def 12: Let C be a curve with "orientation" given by a parametrization $\vec{r}(t)$ with $\vec{r}\,'(t) \neq 0$, $\forall t$. Define the line integral of a vector field $\vec{F}$ along C to be

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds}$$

where $\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ is the unit tangent vector field along C.

(i.e. C is oriented in the direction of $\vec{r}\,'(t)$ or $\hat{T}$ at every point)



Note: If $\vec{r} : [a,b] \to \mathbb{R}^n$ ($n = 2$ or 3) then

$$\int_C \vec{F} \cdot \underbrace{\hat{T}\, ds} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \|\vec{r}\,'(t)\|\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\vec{r}\,'(t)\, dt}_{d\vec{r}}$$

(Abuse of notations: $\vec{r}(t) = \vec{r}(x(t), y(t)) = x(t)\hat{i} + y(t)\hat{j}$ because of the position vector field $\vec{r} = x\hat{i} + y\hat{j}$)

$\therefore$ naturally, we denote $\boxed{d\vec{r} = \hat{T}\, ds}$

and $$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}}$$

<u>eg 38</u> : $\vec{F}(x,y,z) = z\hat{i} + xy\,\hat{j} - y^2\hat{k}$

$C : \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}, \quad 0 \le t \le 1$

<u>Soln</u> : $d\vec{r} = \left(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)dt$

& $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C \left(z(t)\hat{i} + x(t)y(t)\hat{j} - y(t)^2\hat{k}\right)\cdot d\vec{r}$$

$$= \int_0^1 \left(\sqrt{t}\,\hat{i} + t^3\hat{j} - y^2\hat{k}\right)\cdot\left(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)dt$$

$$= \cdots = \frac{17}{20} \quad \text{(check!)}$$

※

<u>In components form</u> :

Line integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$$C : \vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$$

can be expressed as

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_a^b (Mg' + Nh')\,dt$$

(more explicitly: $\int_a^b [M(g(t), h(t))\, g'(t) + N(g(t), h(t))\, h'(t)]\, dt$)

Note that, $\begin{cases} x = g(t) \\ y = h(t) \end{cases}$

$$\Rightarrow \begin{cases} dx = g'(t)\, dt \\ dy = h'(t)\, dt \end{cases}$$

$$\therefore \boxed{\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b M\,dx + N\,dy}$$

Similarly, for 3-dim

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b M\,dx + N\,dy + L\,dz}$$

(for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$)

<u>Another way to justify the notation</u>:

$\vec{r} = (x, y, z)$ the position vector

$\Rightarrow$ $\boxed{d\vec{r} = (dx, dy, dz)}$ (natural notation)

Then $\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C (M, N, L) \cdot (dx, dy, dz)$

$$= \int_C M\,dx + N\,dy + L\,dz.$$

eg 39: Evaluate $I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le 2\pi)$

$= (\cos t, \sin t, t)$

Soln: $I = \int_C (-\sin t)\,d(\cos t) + t\,d(\sin t) + 2\cos t\,dt$

$= \int_0^{2\pi} \left(\sin^2 t + t\cos t + 2\cos t\right) dt$

$= \cdots = \pi$ (check!) ※

$\left(d\vec{r} = (-\sin t, \cos t, 1)\,dt \;\;\&\;\; \vec{r}\,'(t) = (-\sin t, \cos t, 1) \right)$

Physics

(1) $\vec{F}$ = Force field

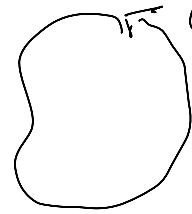
C = oriented curve

then $$\boxed{W = \int_C \vec{F} \cdot \hat{T}\, ds}$$

is the workdone in moving an object along C.

(2) $\vec{F}$ = velocity vector field of fluid

C = oriented curve



C

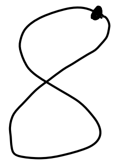
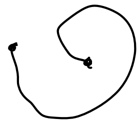
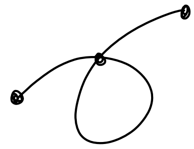
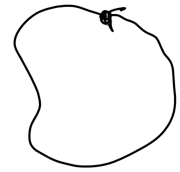
Then $$\boxed{\text{Flow} = \int_C \vec{F} \cdot \hat{T}\, ds}$$

Flow along the curve C.

If C is "closed", the flow is also called a circulation.

$$\boxed{\begin{array}{l}\text{Def 13: A curve is said to be} \\ \text{(i) simple if it does not intersect with itself except possibly at end points.} \\ \text{(ii) closed if starting point = end point.} \\ \text{(iii) simple closed curve if it is both simple and closed.}\end{array}}$$

Note :

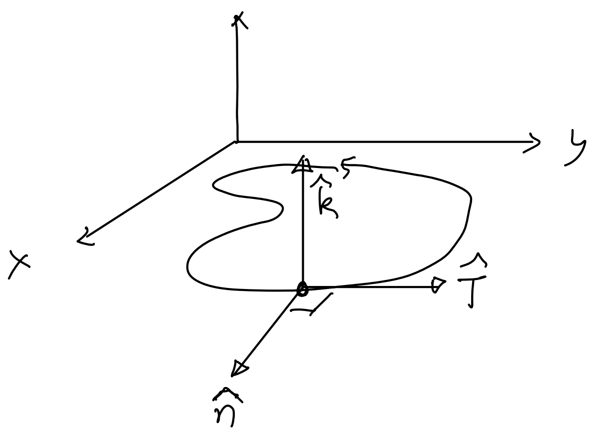
				
simple	NO	Yes	NO	Yes
closed	Yes	NO	NO	Yes

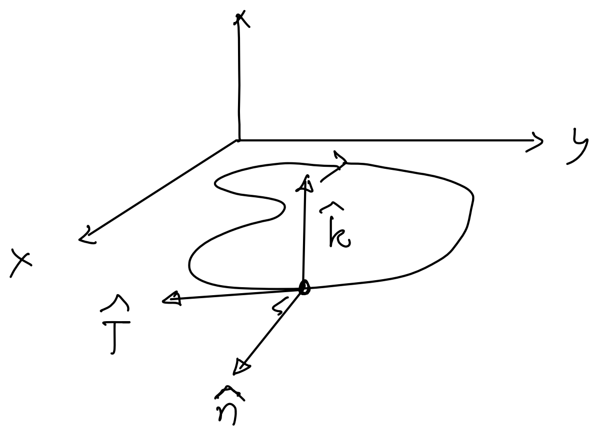
(3) $\vec{F}$ = velocity of fluid

C = oriented plane curve ($C \subset \mathbb{R}^2$) (simple, closed)

with parametrization $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$\hat{n}$ = outward-pointing unit normal (vector) to the curve C





$$\boxed{\hat{n} = \hat{T} \times \hat{k}}$$

if C is of anti-clockwise orientation

$$\boxed{\hat{n} = -\hat{T} \times \hat{k}}$$

if C is of clockwise orientation.

Formula for $\hat{n}$ (wrt the parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$)

Recall $\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{x'(t)\hat{i} + y'(t)\hat{j}}{\|\vec{r}\,'(t)\|}$

(in arc-length parametrization : $\hat{T} = \dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j}$)

Anti-clockwise :

$$\hat{n} = \hat{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{x'}{\|\vec{r}\,'\|} & \frac{y'}{\|\vec{r}\,'\|} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \hat{n} = \frac{y'(t)\hat{i} - x'(t)\hat{j}}{\|\vec{r}\,'(t)\|} \qquad \left(\text{or } \hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \right)$$

Clockwise : $\hat{n} = \dfrac{-y'(t)\hat{i} + x'(t)\hat{j}}{\|\vec{r}\,'(t)\|}$ $\left(\text{or } \hat{n} = -\dfrac{dy}{ds}\hat{i} + \dfrac{dx}{ds}\hat{j} \right)$

$$\boxed{\text{Flux of } \vec{F} \text{ across } C \overset{\text{def}}{=} \int_C \vec{F} \cdot \hat{n}\, ds}$$

(amount of fluid getting out of the closed (plane) curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is anti-clockwise parametrization of C (closed curve)

Then

Flux of $\vec{F}$ across C

$$= \oint_C (M\hat{i} + N\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$$

$$= \oint_C M\,dy - N\,dx$$

<u>Remark</u>:

- $\ointctrclockwise$: curve is <u>closed</u> & in <u>anti-clockwise</u> direction
- $\ointclockwise$ = curve is <u>closed</u> & in <u>clockwise</u> direction

  (not a common notation)
- But in some books, only "$\oint$" is used, <u>NO</u> arrow, Then one needs to determine the orientation from the context.
- <u>Convention</u>: If no orientation is mentioned, "$\oint$" without arrow means <u>anti-clockwise orientation</u> (<u>positive orientation</u>)

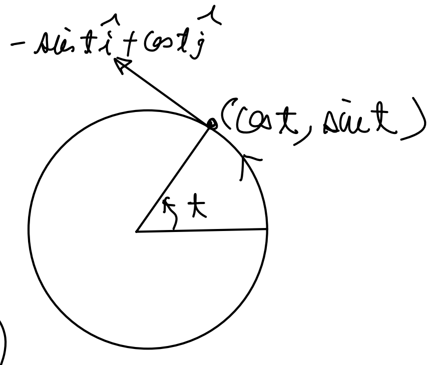
eg 40: Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: x^2+y^2=1$

Find the flow (anti-clockwisely) along C and flux across C.

Soln: Let $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

Note: correct orientation



Then flow $= \oint_C \vec{F}\cdot\hat{T}\,ds$

$= \oint_C \vec{F}\cdot d\vec{r} \quad \left(= \oint_C M\,dx + N\,dy\right)$

$= \int_0^{2\pi} [(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}][-\sin t\,\hat{i} + \cos t\,\hat{j}]\,dt$

$= \int_0^{2\pi} [\sin t(\sin t - \cos t) + \cos^2 t]\,dt$

$= \cdots = 2\pi$ (check!)

flux $= \oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx$

$= \int_0^{2\pi} (\cos t - \sin t)\,d(\sin t) - \cos t\,d\cos t$

$= \int_0^{2\pi} [\cos t(\cos t - \sin t) + \sin t\cos t]\,dt$

$= \cdots = \pi$ (check!)

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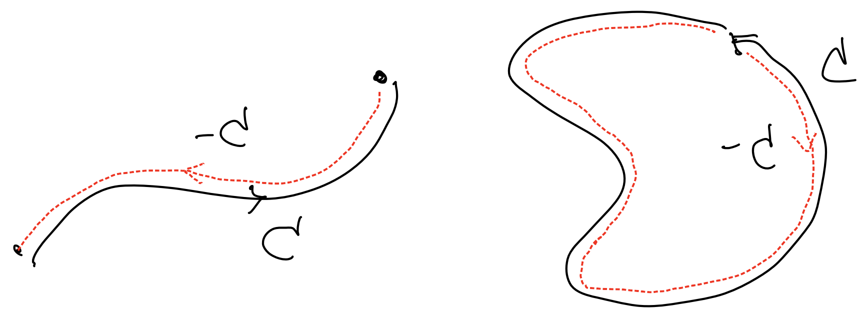
Remark: If C is an oriented curve, then denote by "$-C$" the oriented curve with opposite orientation



- If f is a scalar function

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

as "ds" is not oriented, just "length"

- If $\vec{F}$ is a vector field

flow $$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = -\int_{-C} \vec{F}\cdot\hat{T}\,ds}$$

this $\hat{T}$ is the "$\hat{T}$ for $-C$"

More precise formula:

$$\boxed{\int_C \vec{F}\cdot\hat{T}_C\,ds = -\int_{-C} \vec{F}\cdot\hat{T}_{-C}\,ds}$$

- But for flux $$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_{-C} \vec{F}\cdot\hat{n}\,ds}$$ $\hat{n}$ always outward

Summary:

scalar f	$\int_C f\, ds$ indep. of orientation	ds have no direction
vector $\vec{F}$ flow	$\int_C \vec{F}\cdot\hat{T}\, ds$ depends on orientation	$\hat{T}$ depends on direction
flux	$\int_C \vec{F}\cdot\hat{n}\, ds$ indep. of orientation	$\hat{n}$ always outward