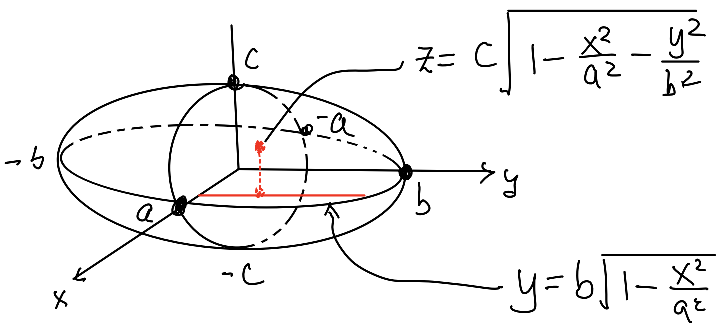
eg18: Volume of Ellipsoid

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c>0)$$

Soln



By symmetry, we can consider the 1st octant only and

$$\begin{aligned} Vol(D) &= 8 \cdot \text{volume of } D \text{ in 1st octant} \\ &= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1 \, dz \, dy \, dx \\ &= \cdots = \frac{4\pi abc}{3} \quad \text{(optional exercise)} \end{aligned}$$

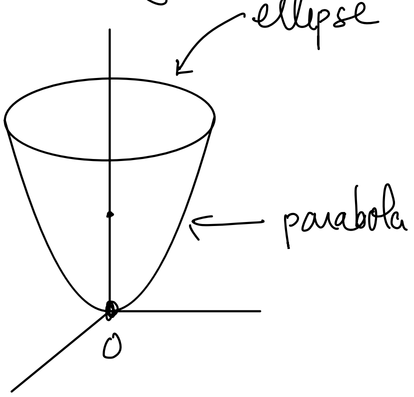
[In fact, we will have a better way to calculate this volume by "change of variables formula" (later)]

eg19: Find the volume of D enclosed by

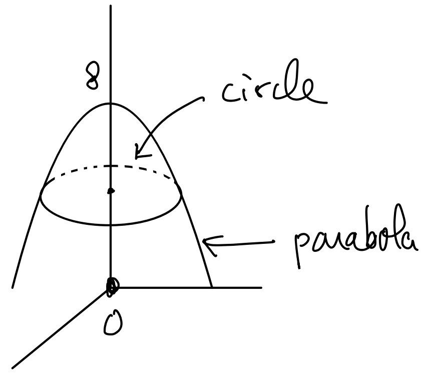
$$z = x^2+3y^2 \quad \text{and} \quad z = 8-x^2-y^2$$

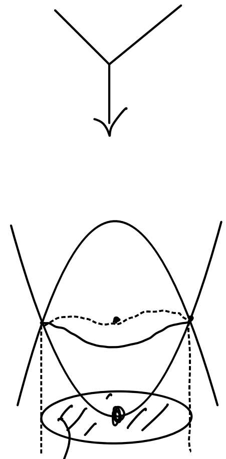
Soln :

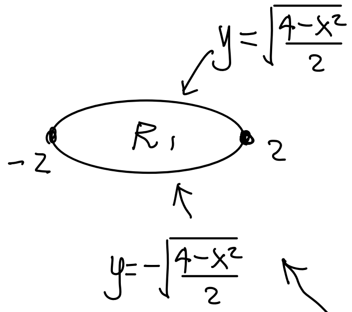
$z = x^2 + 3y^2$



$z = 8 - x^2 - y^2$







R_1

enclosed by the ellipse $x^2+2y^2=4$

These 2 surfaces intersect at the curve given

$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

with projection onto the xy-plane given by

$$x^2 + 2y^2 = 4$$

which is a ellipse.

$$\Rightarrow D = \{ (x,y) \in \underbrace{R_1}_{\{x^2+2y^2\le 4\}} : x^2+3y^2 \le z \le 8 - x^2 - y^2 \}$$

$$= \left\{ -2 \le x \le 2,\ -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}},\ x^2+3y^2 \le z \le 8-x^2-y^2 \right\}$$

$$\therefore \mathrm{Vol}(D) = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} 1 \cdot dz\, dy\, dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3} (4 - x^2)^{\frac{3}{2}} dx \qquad \text{(check!)}$$

$$= 8\pi\sqrt{2} \qquad \text{(Check!)}$$

#

[For those interested, the intersection (space) curve in parameter form is

$$x = 2\cos t,\ y = \sqrt{2}\sin t,\ z = 4 + 2\sin^2 t \qquad (0 \le t \le 2\pi)$$
]

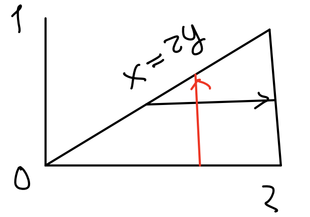
eg 20 Evaluate $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$

Soln:

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$$

$$= \int_0^4 \frac{2}{\sqrt{z}} \left(\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy \right) dz$$

$$= \underbrace{\left(\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy \right)}_{\text{think of this as a double integral over the region}} \left(\int_0^4 \frac{2}{\sqrt{z}}\, dz \right)$$



By Fubini's Thm,

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$$

$$= \left(\int_0^2 \int_0^{\frac{1}{2}x} \cos(x^2)\, dy\, dx \right) \left(\int_0^4 \frac{2}{\sqrt{z}}\, dz \right)$$

$$= \left(\int_0^2 \cos(x^2) \left(\int_0^{\frac{x}{2}} dy \right) dx \right) \left(\int_0^4 \frac{2}{\sqrt{z}} dz \right)$$

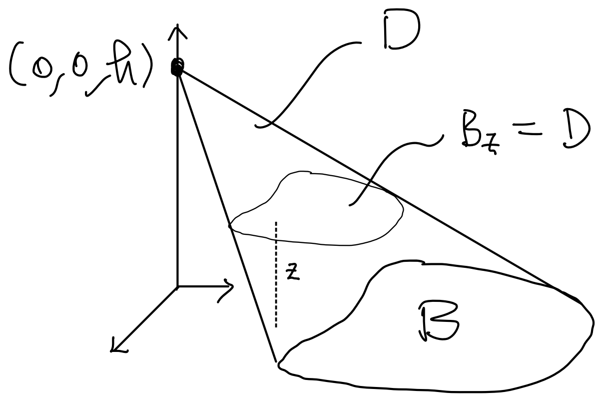
$$= \left(\int_0^2 \frac{x}{2} \cos(x^2) dx \right) \left(\int_0^4 \frac{2}{\sqrt{z}} dz \right)$$

$= 2 \sin 4$ (check!) (using integration-by-parts)

#

e.g.21 Let B (base) be a "nice" subset of $\mathbb{R}^2$.

Let D = cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0,0,h)$ $(h>0)$



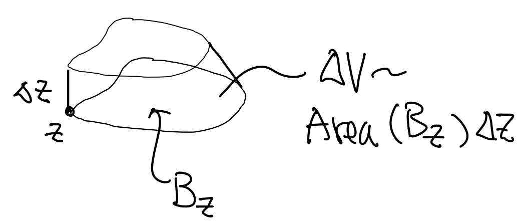
$B_z = D \cap \{\text{horizontal plane at height } z\}$

(B_z "similar" to B)

How to find the volume of D?

Answer: By the concept of Riemann sum

$$Vol(D) = \int_0^h Area(B_z)\, dz$$



ratio of heights: $\frac{h-z}{h} = 1 - \frac{z}{h}$

ratio of areas: $\frac{Area(B_z)}{Area(B)} = \left(1 - \frac{z}{h}\right)^2$ by "similarity"

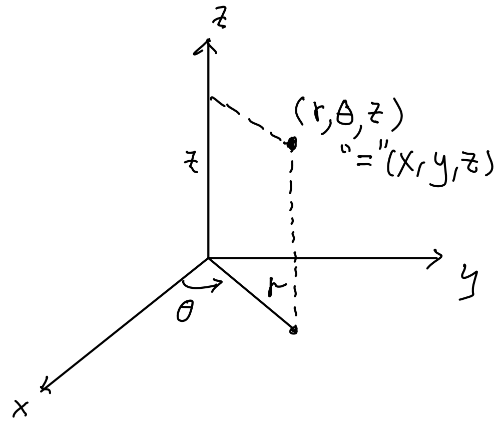
$$\Rightarrow \quad Vol(D) = \int_0^h \left(1-\frac{z}{h}\right)^2 Area(B)\, dz$$

$$= Area(B) \int_0^h \left(1-\frac{z}{h}\right)^2 dz$$

$$= \frac{h}{3} Area(B)$$

※ (check!)

Cylindrical Coordinates in $\mathbb{R}^3$



- (r, θ) = polar coordinates for the xy-plane $(r \geq 0)$

- z = rectangular vertical coordinate

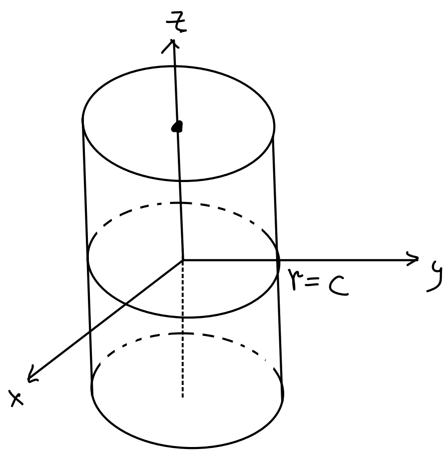
Then a point $P = (x, y, z)$ can be represented by (r, θ, z), where

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

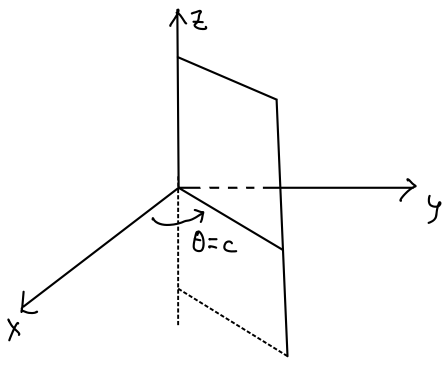
And (r, θ, z) is called the <u>cylindrical coordinates</u> for $\mathbb{R}^3$.

<u>Remark 1</u>: (Let c be a constant)

- $r = c$ $(c > 0)$ describes a <u>cylinder</u>



- $\theta = c$ $(0 \leq c \leq 2\pi)$ describes a vertical half-plane

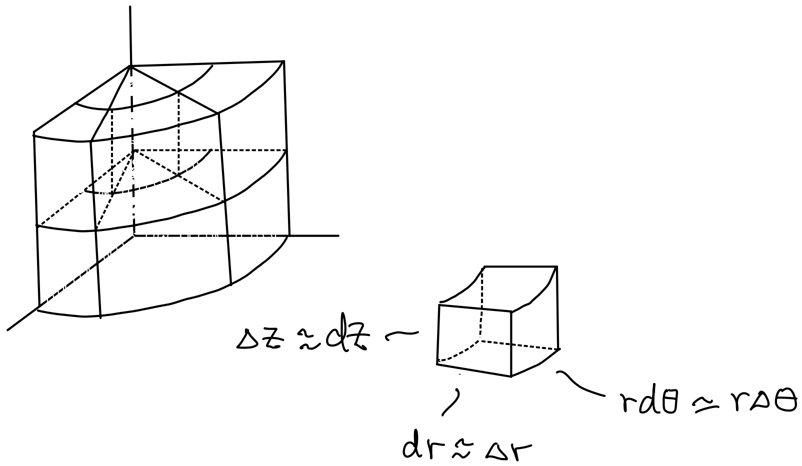


- $z = c$ describes a horizontal plane (as in rectangular coordinates)

Remark 2: We can define cylindrical coordinates in other directions:

e.g. $\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$ (Ex: draw the cylinder $r = c$)

Volume element



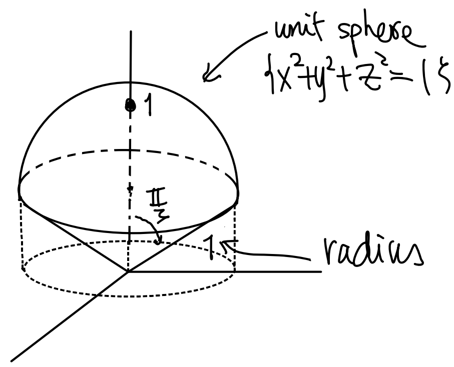
$$dV = \underbrace{dx\,dy}_{\updownarrow}\,\underbrace{dz}_{\updownarrow}$$

$$= r\,dr\,d\theta \cdot dz$$

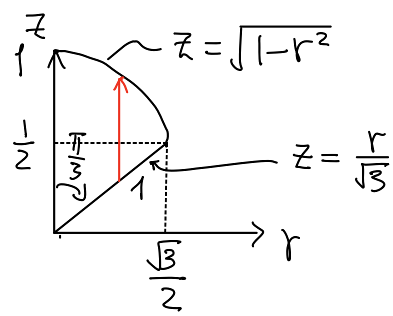
(order of the integration can be changed)

eg22 (see also eg24)

Find the volume of the Ice-cream cone D given as in the figure.



Soln: θ fixed



Fubini's Thm $\Rightarrow$

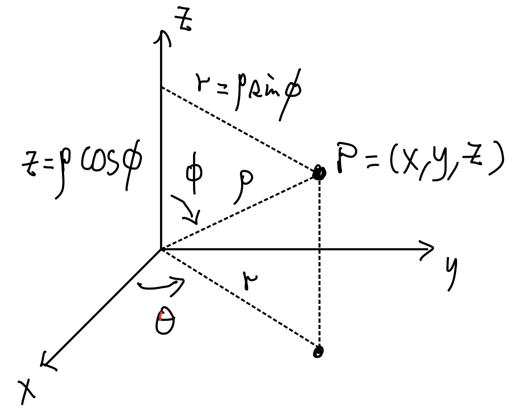
$$Vol(D) = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\left(\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} 1\,dz\right)\underbrace{r\,dr\,d\theta}_{dx\,dy}$$

don't miss this r!

$$= 2\pi\int_0^{\frac{\sqrt{3}}{2}}\left(\sqrt{1-r^2}-\frac{r}{\sqrt{3}}\right)r\,dr = \frac{\pi}{3} \quad \text{(check!)} \quad \#$$

Spherical coordinates in $\mathbb{R}^3$

(ρ, ϕ, θ) where



- ρ = distance from the origin $(\rho \geq 0)$
- ϕ = angle from the positive z-axis to $\overline{OP}$ $\quad (0 \leq \phi \leq \pi)$
- θ = angle from cylindrical coordinate $(0 \leq \theta \leq 2\pi)$

Remark: If (r, θ, z) is the cylindrical coordinates of the point P, then

$$\begin{cases} r = \rho \sin\phi \\ z = \rho \cos\phi \end{cases}$$

(ρ, ϕ can be regarded as polar coordinates of the (z, r) coordinates)

In particular $\quad z^2 + r^2 = \rho^2$.

Then

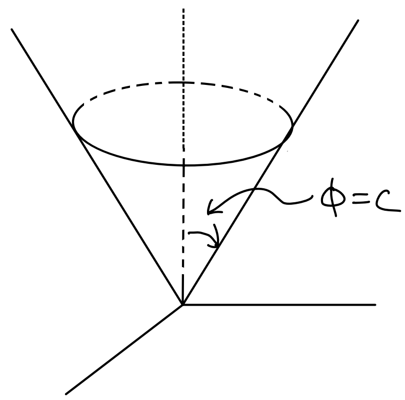
$$\boxed{\begin{array}{ccccc} x & = & r\cos\theta & = & \rho\sin\phi\cos\theta \\ y & = & r\sin\theta & = & \rho\sin\phi\sin\theta \\ z & = & z & = & \rho\cos\phi \end{array}}$$

rectangular $\quad$ cylindrical $\quad$ spherical

Remark: If c is a constant, then

- $\rho = c \quad (c>0)$ describes a sphere of radius c
- $\theta = c$ describes a vertical half-plane.
- $\phi = c$ describes

$$= \begin{cases} +\text{ve } z\text{-axis}, & \text{if } c=0 \\ -\text{ve } z\text{-axis}, & \text{if } c=\pi \\ xy\text{-plane}, & \text{if } c=\frac{\pi}{2} \\ \text{cone}, & \text{otherwise} \end{cases}$$

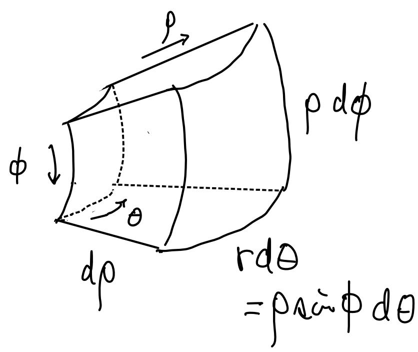


(upward $0<c<\frac{\pi}{2}$
downward $\frac{\pi}{2}<c<\pi$)

Volume element

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho\sin\phi)(\rho\,d\rho\,d\phi)\,d\theta$$



i.e. $\boxed{dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$

eg23 Convert the following into spherical coordinates

(1) $x^2+y^2+(z-1)^2 = 1$ (sphere)

(2) $z = -\sqrt{x^2+y^2}$ (cone)

Soln: (1) Sub.
$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \quad \text{——} (*)$$

into $x^2+y^2+(z-1)^2 = 1$

$$\Rightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Rightarrow \quad \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\Rightarrow \quad \rho^2 = 2\rho\cos\phi$$

$$\Rightarrow \quad \rho = 2\cos\phi \qquad \#$$

(2) Sub (*) into $z = -\sqrt{x^2+y^2}$ $(= -r)$

$$\Rightarrow \quad \rho\cos\phi = -\rho\sin\phi \qquad (\rho \geq 0)$$

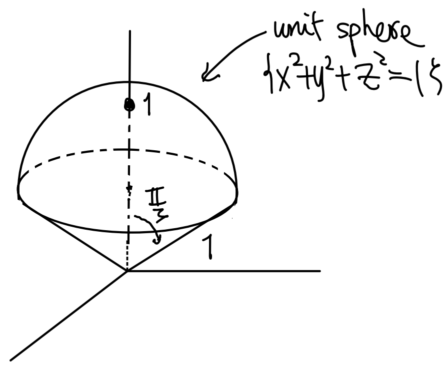
($\rho = 0$ is the point $(0,0,0)$)

& $\rho \neq 0 \Rightarrow \cos\phi = -\sin\phi \quad (0 \leq \phi \leq \pi)$

$$\Rightarrow \quad \phi = \frac{3\pi}{4} \qquad \#$$

eg24 (see eg22)

Volume of ice-cream cone I again, in spherical coordinates



Soln: The ice-cream cone I is given by

$$\left\{ 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \frac{\pi}{3},\ 0 \leq \theta \leq 2\pi \right\}$$

$$\mathrm{Vol}(T) = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 1 \cdot \underbrace{\rho^2 \sin\phi}\, d\rho \, d\phi \, d\theta$$

don't miss this

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}} \sin\phi \, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right)$$

$$= \frac{\pi}{3} \quad \text{(Check!)}$$

eg 25

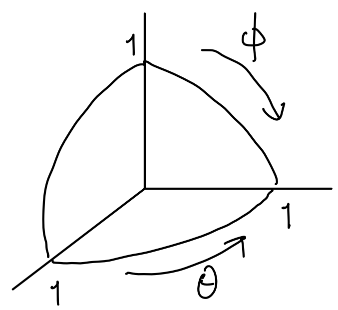
$$f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}, & \text{if } (x,y,z) \neq (0,0,0) \\ 0, & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

(In fact, f is continuous, but it is sufficient to know f is continuous except at the origin $(0,0,0)$)

Let D = unit ball centered at origin intersecting with the 1st octant

Find the average of f over D.

Soln: D can be represented in spherical coordinates:



$$\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \frac{\pi}{2} \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$$

And $f(x,y,z) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \dfrac{\rho^2 \sin^2\phi}{\rho}$ $\quad ((x,y,z) \neq (0,0,0))$

$= \rho \sin^2\phi$ $\quad$ ($\because f \to 0$ as $\rho \to 0 \Rightarrow f$ is cts at 0)

Hence $\iiint_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \underbrace{(\rho \sin^2\phi)}_{\text{function}} \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{volume element}}$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

$= \frac{\pi}{2} \left(\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \right) \left(\int_0^1 \rho^3 \, d\rho \right)$

$= \frac{\pi}{12}$ (check!)

$\text{Vol}(D) = \frac{1}{8} \text{Vol}(\text{unit ball}) = \frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}$

$\Rightarrow$ Average of f over $D = \dfrac{1}{\text{Vol}(D)} \iiint_D f(x,y,z)\,dV$

$= \frac{1}{2}$ $\quad$ #

eg 26: (Improper integrals)

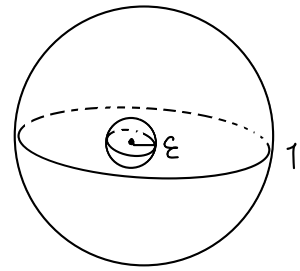
Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$ $\quad$ (both unbounded as $\rho \to 0$)

$g(x,y,z) = \dfrac{1}{(\sqrt{x^2+y^2+z^2})^3} = \dfrac{1}{\rho^3}$

over unit ball $B = \{ (\rho, \theta, \phi) : 0 \le \rho \le 1 \}$

(i) Does $\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV$ exist?

where $B_\varepsilon = \{(\rho,\phi,\theta) : 0 \le \rho \le \varepsilon\}$



(ii) Does $\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV$ exist?

Answer:

(i)
$$\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV = \lim_{\varepsilon\to 0} \int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^2}\cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$
$$= \lim_{\varepsilon\to 0} 2\pi \left(\int_0^\pi \sin\phi\, d\phi\right)\left(\int_\varepsilon^1 d\rho\right)$$
$$= \lim_{\varepsilon\to 0} 4\pi(1-\varepsilon) = 4\pi$$

(ii)
$$\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV = \lim_{\varepsilon\to 0} \int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^3}\cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$
$$= \lim_{\varepsilon\to 0} 4\pi \left(\int_\varepsilon^1 \frac{d\rho}{\rho}\right) = \lim_{\varepsilon\to 0} 4\pi \ln\frac{1}{\varepsilon}$$

Doesn't exist #

Terminology: • $f = \frac{1}{\rho^2}$ is said to be "integrable" over B

(in the sense of improper integral)

• $g = \frac{1}{\rho^3}$ is said to be "non integrable" over B

Question: determine all $\beta > 0$ such that

$$f = \frac{1}{\rho^\beta} \text{ is "integrable" over } B \subset \mathbb{R}^3$$

Similar question in $\mathbb{R}^2$: determine all $\beta > 0$ such that

$$f = \frac{1}{r^\beta} \text{ is "integrable" in } \{r \leq 1\} \subset \mathbb{R}^2$$

(even in $\mathbb{R}^1$: $f = \frac{1}{|x|^\beta}$)

Application of Multiple Integrals

In applications, we often use the following:

In 2-dim: Let R be a region in $\mathbb{R}^2$ with density $\delta(x,y)$

- First moment about y-axis: $M_y = \iint_R x\,\delta(x,y)\,dA$
- First moment about x-axis: $M_x = \iint_R y\,\delta(x,y)\,dA$
- Mass: $M = \iint_R \delta(x,y)\,dA$
- Center of Mass (Centroid)

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

In 3-dim, D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

- First moment:
  - about yz-plane, $M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$
  - about xz-plane, $M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$
  - about xy-plane, $M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$
- Mass: $M = \iiint_D \delta(x,y,z)\,dV$
- Center of Mass (Centroid) $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

In 2-dim, R = region in $\mathbb{R}^2$ with density $\delta(x,y)$

<u>Moments of inertia</u>

- about x-axis : $I_x = \iint_R y^2 \delta(x,y)\,dA$
- about y-axis : $I_y = \iint_R x^2 \delta(x,y)\,dA$
- about line L : $I_L = \iint_R r(x,y)^2 \delta(x,y)\,dA$

  where $r(x,y)$ = distance between (x,y) and L.

- about the origin : $I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$

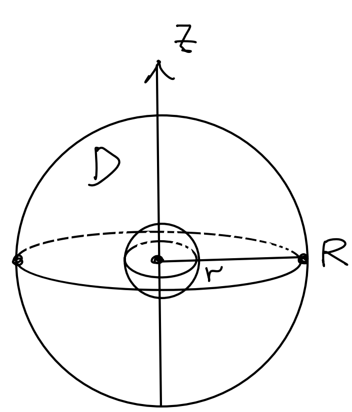
In 3-dim, D = solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

<u>Moments of Inertia</u>

- around x-axis : $I_x = \iiint_D (y^2+z^2)\delta(x,y,z)\,dV$
- around y-axis : $I_y = \iiint_D (x^2+z^2)\delta(x,y,z)\,dV$
- around z-axis : $I_z = \iiint_D (x^2+y^2)\delta(x,y,z)\,dV$
- around Line L : $I_L = \iiint_D r(x,y,z)^2 \delta(x,y,z)\,dV$

  where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 27 : Consider D : $r^2 \le x^2+y^2+z^2 \le R^2$
$(0 < r < R)$



with density $\delta(x,y,z) \equiv \delta$
(constant density function, i.e. uniform mass)

Express I_z in term of m = Mass of D, r and R.

Soln: $I_z \stackrel{\text{def}}{=\!=} \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$

$$= \delta \iiint_D (x^2+y^2)\,dV = \delta \int_0^{2\pi}\int_0^{\pi}\int_r^{R} (\rho\sin\phi)^2 \cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \delta \cdot 2\pi \left(\int_0^{\pi} \sin^3\phi\,d\phi\right)\left(\int_r^{R} \rho^4\,d\rho\right)$$

$$= \frac{8\pi}{15}(R^5-r^5)\,\delta \qquad \text{(check!)}$$

$$m = \text{Mass} = \iiint_D \delta(x,y,z)\,dV = \delta \iiint_D dV = \delta\,\mathrm{Vol}(D)$$

$$= \delta \cdot \frac{4\pi}{3}(R^3-r^3)$$

$$\therefore \quad \boxed{I_z = \frac{2m}{5}\,\frac{R^5-r^5}{R^3-r^3}}$$

Observation: Two limiting cases:

(i) $r \to 0$, i.e. the whole solid ball $\quad \boxed{I_z = \frac{2m}{5}R^2}$

(ii) $r \to R$, i.e. a (hollow) sphere made of "infinitesimally" thin sheet:

$$I_z = \lim_{r \to R} \frac{2m}{5} \cdot \frac{R^5 - r^5}{R^3 - r^3} = \frac{2m}{5} \cdot \frac{5R^4}{3R^2} \quad \text{(check!)}$$

$$\therefore \quad \boxed{I_z = \frac{2m}{3} R^2}$$

Moment of inertia of the <u>hollow sphere</u>

> moment of inertia of the <u>solid ball</u>

(assuming the same (uniform) mass m) ※