eg5 : let $R=[0,1]\times[0,1]$

$$f(x,y)=\begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is not integrable over R. (using (ii))

Soln : $\forall$ partition P of $R=R_1\cup\cdots\cup R_n$ (R_k subrectangles)

One can find points $(x_k, y_k)\in R_k$, $\forall k$, such that

both x_k, y_k are rational. (why?)

The corresponding Riemann sum equals

$$S_n(f,P)=\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k=\sum_{k=1}^{n} 0\cdot\Delta A_k=0$$

On the other hand, one can find points $(x_k', y_k')\in R_k$, $\forall k$, such that at least one of the x_k', y_k' is irrational. (why?)

The corresponding Riemann sum equals

$$S_n'(f,P)=\sum_{k=1}^{n} f(x_k',y_k')\,\Delta A_k=\sum_{k=1}^{n} 1\cdot\Delta A_k=1$$

Since $S_n(f,P)\to 0 \neq 1 \leftarrow S_n'(f,P)$

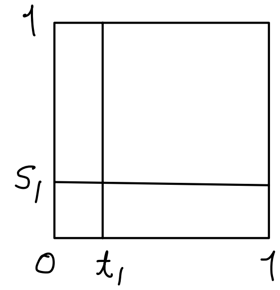
f is not integrable. ※

eg6 : Let $R=[0,1]\times[0,1]$

$$f(x,y)=\begin{cases} \frac{1}{xy}, & \text{if } x\neq 0 \text{ and } y\neq 0 \\ 0, & \text{if } x=0 \text{ or } y=0 \end{cases} \quad (\geq 0 \text{ on } R)$$

Then f is not integrable over R. (using (i))

<u>Soln</u> In any partition P of R, there is a subrectangle $R_1 = [0, t_1] \times [0, s_1]$



Choose $(x_1, y_1) = (t_1^2, s_1^2) \in R_1$ $\quad (0 < t_1^2 < t_1 < 1,\ 0 < s_1^2 < s_1 < 1)$

Then Riemann sum

$$S(f,P) = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = f(x_1,y_1)\Delta A_1 + \underbrace{\sum_{k=2}^{n} f(x_k,y_k)\Delta A_k}_{\geq 0}$$

$$\geq \frac{1}{t_1^2 s_1^2} t_1 s_1 = \frac{1}{t_1 s_1}$$

Since $0 < t_1, s_1 \leq \|P\| \to 0$, $\quad t_1 \,\&\, s_1 \to 0$

Hence $S(f,P) \geq \frac{1}{t_1 s_1} \to \infty$ as $\|P\| \to 0$

$\therefore$ Limit doesn't exist, & f is not integrable ※

<u>Remark</u>: Egs 5 & 6 show that we need "<u>condition(s)</u>" to ensure the <u>integrability</u> of a function over closed (& bdd) rectangle.

> <u>Prop1</u>: Let $R = [a,b] \times [c,d]$ be a <u>closed (& bdd) rectangle</u>, and $f(x,y)$ be an <u>integrable</u> function over R, then f is <u>bounded</u> on R.
> (i.e. $\exists\, M > 0$ such that "$|f(x,y)| \leq M,\ \forall (x,y) \in R$.")

<u>Pf</u>: Omitted (eg6 above gives an idea of proof.)

Remark: From eg 5, "boundedness" is necessary, but not sufficient for integrability.

$$\boxed{\begin{array}{c}\text{integrable} \Rightarrow \text{bounded} \\ \not\Leftarrow \\ (\leftarrow \text{ in general})\end{array}}$$

$$\boxed{\begin{array}{l}\text{Prop 2: Let } R=[a,b]\times[c,d] \text{ be a } \underline{\text{closed (\& bdd) rectangle}}, \\ \text{and } f(x,y) \text{ be a } \underline{\text{continuous}} \text{ function on } R, \\ \text{then } f \text{ is } \underline{\text{integrable}} \text{ on } R.\end{array}}$$

Pf: Omitted (See proof in 1-variable case in MATH 2060 for an idea of proof.)

Remarks: (i) Note that a continuous function on closed (& bdd) rectangle is always bounded (Props 1 & 2 are consistent) (MATH 2050 for 1-variable situation)

(ii) "continuity" (on closed (& bdd) rectangle) is sufficient, but not necessary. In fact, Prop 2 can be generalized to a bounded function on a closed rectangle with a "small" set of discontinuity. The precise concept is "measure zero set" (MATH 4050 Real Analysis). For us, we have

Prop 2' For function over closed (& bdd) rectangle

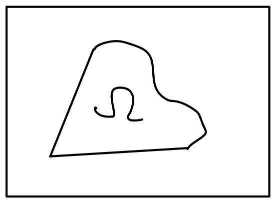
(a) bounded + "continuous except finitely many points"

$\Longrightarrow$ integrable.

(b) bounded + "continuous except finitely many differentiable curves"

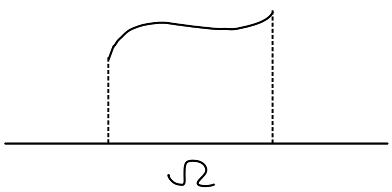
$\Rightarrow$ integrable

egs (i)



$$f(x,y) = \begin{cases} \text{continuous on } \Omega \text{ (and bounded)} \\ 0, \quad \text{otherwise (outside } \Omega) \end{cases}$$

(ii)



Furthermore, we have

Prop 3: Let $R = [a,b] \times [c,d]$ be a closed (& bdd) rectangle,

$f(x,y)$ and $g(x,y)$ be functions on R, and

$k \in \mathbb{R}$ is a constant.

(1) If f & g are integrable over R, then $f \pm g$ and kf are integrable over R.

(2) In the case of (1), we have

$$\iint_R [f \pm g](x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

and $$\iint_R k f(x,y)\,dA = k \iint_R f(x,y)\,dA.$$

<u>Pf</u>: Omitted (Obvious from the concept of Riemann sum.)

<u>Remark</u>: This Prop 3 implies that the set of integrable functions over (fixed) R forms a "vector space over $\mathbb{R}$", & "(double) integral" is <u>linear</u>.