A sufficient condition for differentiability:

Thm Let $\Omega \subseteq \mathbb{R}^n$ be open, f be C^1 on Ω, then f is differentiable on Ω

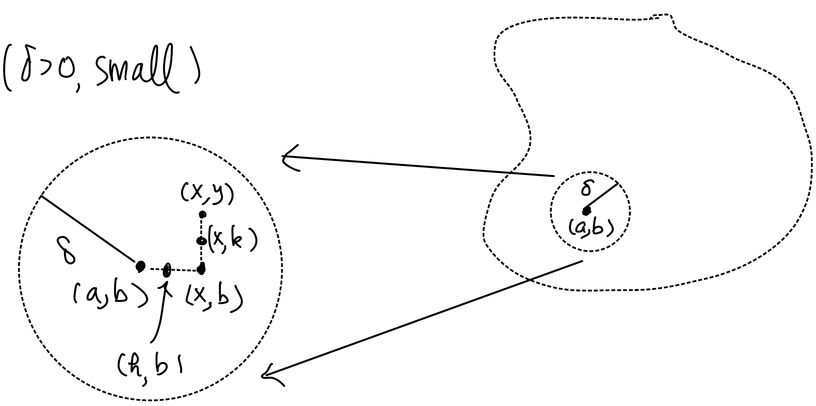
(The assumption requires all $\frac{\partial f}{\partial x_i}$ exist on Ω, not just at a single pt. $\vec{a}$)

Pf: (We prove it for 2-variables, similar proof for general case)

Suppose $(a,b) \in \Omega$

& $B_\delta(a,b) \subset \Omega$ ($\delta > 0$, small)

For any $(x,y) \in B_\delta(a,b)$



$$f(x,y) - f(a,b) = \underbrace{f(x,y) - f(x,b)} + \underbrace{f(x,b) - f(a,b)}$$

$$= f_y(x,k)(y-b) + f_x(h,b)(x-a) \quad \text{(by Mean Value Theorem)}$$

where k between y & b ; h between x & a.

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} = \frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{|f_y(x,k)(y-b) + f_x(h,b)(x-a) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{\left|\left(f_x(h,b)-f_x(a,b)\right)(x-a)+\left(f_y(x,k)-f_y(a,b)\right)(y-b)\right|}{\|(x,y)-(a,b)\|}$$

$$\begin{pmatrix}\text{Cauchy-}\\ \text{Schwarz}\end{pmatrix} \le \frac{\sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2+\left(f_y(x,k)-f_y(a,b)\right)^2}\ \sqrt{(x-a)^2+(y-b)^2}}{\|(x,y)-(a,b)\|}$$

$$= \sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2+\left(f_y(x,k)-f_y(a,b)\right)^2}$$

Note that if $(x,y)\to(a,b)$, then $(h,k)\to(a,b)$.

Hence

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} \le \sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2+\left(f_y(x,k)-f_y(a,b)\right)^2} \to 0$$

as $(x,y)\to(a,b)$

because f_x & f_y are continuous.

$\therefore$ f is differentiable at (a,b).

Since $(a,b)\in\Omega$ is arbitrary, f is differentiable on Ω #

eg s: (1) constant functions $f(\vec{x})=c$ is differentiable

(2) coordinate functions $f(\vec{x})=x_i$ are differentiable

(3) (1)&(2) $\Rightarrow$ $f(\vec{x}) = a+b_1x_1+\cdots+b_nx_n$ is differentiable

(Linear function. For this f, what is the linearization $L(\vec{x})$ at $\vec{x}=\vec{0}$)

(4) Polynomials & rational functions are differentiable (in their domain of definition).

(5) If $f(\vec{x})$ is differentiable, then $e^{f(\vec{x})}$, $\sin(f(\vec{x}))$, $\cos(f(\vec{x}))$ are differentiable.

And $\left.\begin{array}{ll} \ln(f(\vec{x})) & \text{when } f(\vec{x})>0 \\ \sqrt{f(\vec{x})} & \text{when } f(\vec{x})>0 \\ |f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \\ \ln|f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \end{array}\right\}$ are differentiable.

In particular, for example $\dfrac{e^{\sqrt{4+\sin(x^2+xy)}}}{\ln(1+\cos(x^2y))}$ is differentiable in the domain of definition

eg: $f(x,y,z) = xe^{x+y} - \ln(x+z)$ $\left(= xe^{x+y} - \log(x+z)\right)$

Domain of $f = \{(x,y,z)\in\mathbb{R}^3 : x+z>0\}$ (is open)

$$\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y} - \frac{1}{x+z}$$

$$\frac{\partial f}{\partial y} = xe^{x+y}$$

$x+z>0$ all terms are continuous in the domain of f.

$$\frac{\partial f}{\partial z} = -\frac{1}{x+z}$$

$\Rightarrow$ f is C^1 (on its domain) $\Rightarrow$ f is differentiable (on its domain).

Gradient and Directional Derivative

Def: Let
- $f: \Omega \to \mathbb{R}$, $(\Omega \subseteq \mathbb{R}^n$, open$)$
- $\vec{a} \in \Omega$

Then the gradient vector of f at $\vec{a}$ is defined to be

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \cdots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Remark: Using $\vec{\nabla} f$, linearization of f at $\vec{a}$ can be written as

$$\boxed{\begin{aligned} L(\vec{x}) &= f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \\ &= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) \end{aligned}}$$

eg: $f(x,y) = x^2 + 2xy$

$$\frac{\partial f}{\partial x} = 2x + 2y, \quad \frac{\partial f}{\partial y} = 2x$$

$$\therefore \vec{\nabla} f(x,y) = (2x+2y, 2x)$$

(eg. $\vec{\nabla} f(1,2) = (6,2)$)

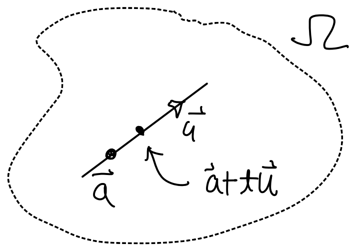
Def: Let $\begin{cases} \bullet\ f : \Omega \to \mathbb{R}, & (\Omega \subseteq \mathbb{R}^n, \text{ open}) \\ \bullet\ \vec{a} \in \Omega \\ \bullet\ \vec{u} \in \mathbb{R}^n \text{ be a } \underline{\text{unit}} \text{ vector, i.e. } \|\vec{u}\| = 1. \end{cases}$

Then the <u>directional derivative</u> of f in the direction of $\vec{u}$ at $\vec{a}$ is defined to be

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

(= rate of change of f in the direction of $\vec{u}$ at the point $\vec{a}$)

<u>Remark</u>: If $\vec{u} = (0, \cdots, \underset{j\text{th component}}{1}, \cdots, 0) = \vec{e}_j$, $\quad j = 1, \cdots, n$



$$D_{\vec{e}_j} f(\vec{a}) = \frac{\partial f}{\partial x_j}(\vec{a})$$

Thm Suppose f is <u>differentiable</u> at $\vec{a}$.
Let $\vec{u}$ be a <u>unit</u> vector in $\mathbb{R}^n$, then
$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

<u>eg</u>: Let $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$.
Find the rate of change of f at $(1, \sqrt{2})$ in the direction of $\vec{v} = (1, -1)$ (not necessary unit).

<u>Remark</u>: $\vec{v} \neq \vec{0} \in \mathbb{R}^n$, not necessary unit, then
the <u>direction of $\vec{v}$</u> is $\frac{\vec{v}}{\|\vec{v}\|}$ (a unit vector).

<u>Soln</u>: Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{1^2+(-1)^2}}(1,-1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\sin^{-1}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\sin^{-1}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{-x}{y^2}$$

[Note, $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous "near" $(1,\sqrt{2}) \Rightarrow f$ is C^1 near $(1,\sqrt{2})$]
f is differentiable at $(1, \sqrt{2})$

$$\overset{\text{Thm}}{\Longrightarrow} \quad D_{\vec{u}} f(1,\sqrt{2}) = \vec{\nabla} f(1,\sqrt{2}) \cdot \vec{u}$$

$$= \left(\frac{\partial f}{\partial x}(1,\sqrt{2}),\ \frac{\partial f}{\partial y}(1,\sqrt{2}) \right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$$

$$= \cdots = \frac{1}{\sqrt{2}} + \frac{1}{2} \quad \text{(Check!)}$$

Pf: (<u>Differentiable $\Rightarrow$ $D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$</u>)

Let $L(\vec{x})$ be the linearization of $f(\vec{x})$ at $\vec{a}$

& $f(\vec{x}) = L(\vec{x}) + \mathcal{E}(\vec{x})$

$$= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \mathcal{E}(\vec{x})$$

with $\dfrac{|\mathcal{E}(\vec{x})|}{\|\vec{x} - \vec{a}\|} \to 0$ as $\vec{x} \to \vec{a}$.

Putting $\vec{x} = \vec{a} + t\vec{u}$, we have

$$f(\vec{a} + t\vec{u}) - f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot t\vec{u} + \mathcal{E}(\vec{a} + t\vec{u})$$

$$\Rightarrow \quad \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla} f(\vec{a}) \cdot \vec{u} + \frac{\mathcal{E}(\vec{a} + t\vec{u})}{t}$$

Note that $|t| = \|\vec{x} - \vec{a}\|$,

$$\left| \frac{\mathcal{E}(\vec{a} + t\vec{u})}{t} \right| = \frac{|\mathcal{E}(\vec{a} + t\vec{u})|}{\|\vec{x} - \vec{a}\|} \to 0 \text{ as } t \to 0 \quad (\vec{x} = \vec{a} + t\vec{u})$$

$$\therefore \quad D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

\#

Geometric Meanings of Gradient $\vec{\nabla} f$

> At a point $\vec{a}$, f increases (decreases) most rapidly in the direction of $\vec{\nabla} f(\vec{a})$ $(-\vec{\nabla} f(\vec{a}))$ at a rate of $\|\vec{\nabla} f(\vec{a})\|$

Idea: If f is differentiable at $\vec{a}$, then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u} \qquad (\text{for } \|\vec{u}\| = 1)$$

Cauchy-Schwarz $\Rightarrow$

$$|D_{\vec{u}} f(\vec{a})| \leq \|\vec{\nabla} f(\vec{a})\| \, \|\vec{u}\|$$

$$= \|\vec{\nabla} f(\vec{a})\|$$

i.e. $$-\|\vec{\nabla} f(\vec{a})\| \leq D_{\vec{u}} f(\vec{a}) \leq \|\vec{\nabla} f(\vec{a})\|$$

"=" holds $\Leftrightarrow \vec{u} = -\dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

"=" holds $\Leftrightarrow \vec{u} = \dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

#

Remark: $D_{\vec{v}} f(\vec{a})$ can be defined for any vector $\vec{v}$, not necessary $\|\vec{v}\| = 1$ and could be $\vec{0}$, by the same definition

$$D_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

One can show that

$$D_{\vec{v}} f(\vec{a}) = \begin{cases} \|\vec{v}\| \, D_{\frac{\vec{v}}{\|\vec{v}\|}} f(\vec{a}), & \text{if } \vec{v} \neq \vec{0} \\ 0 & , \text{if } \vec{v} = \vec{0} \end{cases}$$

and that

$$D_{\vec{v}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{v} \qquad \text{if } f \text{ is differentiable at } \vec{a}$$

(not true in general, if f is not differentiable)
eg $f(x,y) = \sqrt{|xy|}$ at $(0,0)$

Properties of Gradient

If
- $f, g : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open) are differentiable,
- c is a constant,

then

(1) $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$,

(2) $\vec{\nabla}(cf) = c \vec{\nabla} f$

(3) $\vec{\nabla}(fg) = g \vec{\nabla} f + f \vec{\nabla} g$

(4) $\vec{\nabla}\left(\frac{f}{g}\right) = \dfrac{g \vec{\nabla} f - f \vec{\nabla} g}{g^2}$ provided $g \neq 0$

(Pf: Easily from properties of partial derivatives)