Homework 2 for MATH5070
Topology of Manifolds
Due Wednesday, Oct. 2

1. Let $\mathcal{M}_n$ be the space of all $n \times n$ real matrices and Sym_n be the space of all $n \times n$ symmetric matrics. Consider the map

$$f : \mathcal{M}_n \to \mathrm{Sym}_n,\ A \mapsto f(A) = A^t A.$$

(i) Since both $\mathcal{M}_n$ and Sym_n are linear spaces, we can identify $T_A\mathcal{M}_n$ with $\mathcal{M}_n$ and $T_{f(A)}\mathrm{Sym}_n$ with Sym_n. Show that $df_A(B) = A^t B + B^t A$.

(ii) Prove that $I_n \in \mathrm{Sym}_n$ is a regular value of f.

(iii) Conclude that $O(n)$ is a $\frac{n(n-1)}{2}$ dimensional submanifold of $\mathcal{M}_n$.

(iv) Find all regular points, critical points, regular values and critical values of f.

(v) Check Sard's theorem for this example.

2. The Whitney embedding theorem says that if M is an n-dimensional manifold, then there exists an embedding $\iota : M \to \mathbb{R}^{2n}$, i.e., every n-dimensional manifold is diffeomorphic to a submanifold of Euclidean $2n$ dimensional space. We prove an easier version of the theorem.

Theorem Let M be a compact manifold. Then M can be embedded in some Euclidean space.

Hint: Let $\mathcal{A} = \{(\varphi_i, U_i, V_i),\ i = 1, \cdots, r\}$ be an atlas. Let $\{\rho_i,\ i = 1, \cdots, r\}$ be a partition of unity subordinate to this atlas. For each i, let $\psi_i : M \to \mathbb{R}^n$ be the map

$$\psi_i(p) = \begin{cases} \rho_i(p)\varphi_i(p) & \text{if } p \text{ is in } U_i \\ 0 & \text{if } p \text{ is not in } U_i. \end{cases}$$

Show that the map

$$\iota : M \to \mathbb{R}^{nr+r}$$

which maps $p \in M$ to the $(nr + r)$-tuple

$$(\psi_1(p), \cdots, \psi_r(p), \rho_1(p), \cdots \rho_r(p))$$

is an embedding.

3. (Optional) Suppose that v and w are two non-vanishing smooth vector fields which are pointwise proportional, that is, $w = f \cdot v$ for some non-vanishing smooth function f on M. Prove that the respective maximal integral curves $\gamma : I \to M$ and $\tilde{\gamma} : J \to M$ of v and w, both passing through $p \in M$ at time 0, are related by a unique diffeomorphism $F : J \to I$ that preserves 0, i.e., $\tilde{\gamma} = \gamma \circ F$.

In other words, this statement implies that the trajectories of the integral curves depend only on the direction of the underlying vector field, and not on the specific magnitude (as encoded by the function f). The integral curves are then "the same" up to a reparametrization.

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