MATH5021 THEORY OF PDE I
Homework 3

Due Nov 7, 2024, in hard copies. Late assignment is not accepted.

1. [7 points] Let I be a time interval, and let $u : I \to \mathbb{R}^+$ be a locally continuous function satisfying

$$u(t) \leq A + \epsilon F(u(t)) + B[u(t)]^\theta$$

for some $A, B, \epsilon > 0$ and $\theta \in (0,1)$. Here, $F : \mathbb{R}^+ \to \mathbb{R}^+$ is locally bounded. Suppose also that $u(t_0) \leq A'$ for some $t_0 \in I$ and $A' > 0$. Show that if ϵ is sufficiently small depending on A, A', B, θ, F, then we have

$$u(t) \lesssim_\theta A + B^{1/1-\theta}, \quad \forall t \in I.$$

2. [18 points] In this problem, we investigate the semi-linear heat equation. Let $u = u(t,x), x \in \mathbb{R}^n$ be a real-valued function satisfying

$$\partial_t u - \Delta u = f(u), \quad \text{in } [0,\infty) \times \mathbb{R}^n, \quad u(0,x) = u_0(x),$$

Do the following:

a. [5 points] Assume $f(u) = 0$. Let $t > 0$ be fixed. Prove that for each fixed integer $k \geq 0$,

$$\|u(t,\cdot)\|_{H^k(\mathbb{R}^n)} \leq \|u_0\|_{H^k(\mathbb{R}^n)}.$$

b. [7 points] Let $k > \frac{n}{2} + 1$, and

$$f(u) = (\nabla_x u) \cdot (\nabla_x u).$$

Prove that for $T > 0$ chosen sufficiently small,

$$\|u(t)\|_{H^k(\mathbb{R}^n)} \leq \mathfrak{C}, \quad \forall t \in [0,T]$$

holds, where $\mathfrak{C}$ is a constant depends on $\|u_0\|_{H^k(\mathbb{R}^n)}$ and T.

c. [6 points] Assume further

$$\|\nabla_x u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(1+t)^{\frac{1}{2}+\delta_0}}, \quad \text{for some } \delta_0 > 0.$$

Then show that the inequality

$$\|u(t,\cdot)\|_{H^k(\mathbb{R}^n)} \leq C' \|u_0\|_{H^k(\mathbb{R}^n)}$$

holds for all $t > 0$, where C' is a generic constant.