Chapter 1.   Integration on measure spaces.

§1.1   Measurable spaces and measurable functions.

Let $X \neq \emptyset$.   Let $\mathcal{P}_X$ be the power set of X, i.e.

$$\mathcal{P}_X = \{ Y : Y \subset X \}.$$

Def. (σ-algebra) $M \subset \mathcal{P}_X$ is called a σ-algebra on X if

(i) $X \in M$;

(ii) If $A \in M$, then $A^c \in M$ $(A^c := X \backslash A)$

(iii) If $A_R \in M$, $R \geq 1$,   then $\bigcup_{R=1}^{\infty} A_R \in M$.

Remark: (i)-(iii) imply that

· $\emptyset \in M$.

· If $A_R \in M$, $k \geq 1$,   then $\bigcap_{k=1}^{\infty} A_R \in M$.

$$\left(\text{using} \quad \bigcap_{R=1}^{\infty} A_R = \left(\bigcup_{R=1}^{\infty} A_R^c \right)^c \right)$$

Hence a σ-algebra is closed under countable union, intersection and complement.

Example : $\cdot$ $\mathcal{P}_X$

$\cdot$ $\{\emptyset, X\}$

Example :   Let $S \subset \mathcal{P}_X$.

Define
$$M(S) = \bigcap \text{ all } \sigma\text{-algebras on } X \text{ containing } S.$$

We call $M(S)$ the smallest σ-algebra containing S.

(or the σ-algebra generated by S).

Example :   Let X be a topological space.

Let β_X be the σ-algebra on X generated by
the class of open sets in X.

We call β_X the Borel σ-algebra on X.

Def.   A pair (X, M) is said to be a measurable space
if M is a σ-algebra on X.   (可测空间)

Def: A function $f: X \to \mathbb{R}$ is said to be measurable
if $f^{-1}(G) \in M$, $\forall$ open $G \subset \mathbb{R}$.

Remark: Equivalently, f is measurable if

$$f^{-1}(a, b) \in M, \quad \forall \ a, b \in \mathbb{R}, \ a < b.$$

(Using the fact that every open set in $\mathbb{R}$ is
the countable union of finite open intervals)

Prop 1.1. $f: X \to \mathbb{R}$ is measurable iff one of following
properties holds:

① $f^{-1}(a, b) \in M, \quad \forall \ a, b \in \mathbb{R}, \ a < b.$

② $f^{-1}(a, \infty) \in M, \quad \forall \ a \in \mathbb{R}.$

③ $f^{-1}[a, \infty) \in M, \quad \forall \ a \in \mathbb{R}$

④ $f^{-1}(-\infty, a) \in M, \quad \forall \ a \in \mathbb{R}$

⑤ $f^{-1}(-\infty, a] \in M, \quad \forall \ a \in \mathbb{R}.$

Pf.   WLOG, we prove

$\quad\quad$ f is measurable $\iff$ ② holds.

Clearly, "$\implies$" holds.

Now to prove "$\impliedby$", suppose ② holds.

We want to prove ① holds.

Notice that

$$f^{-1}[a, \infty) = \bigcap_{n=1}^{\infty} f^{-1}\left(a - \tfrac{1}{n}, \infty\right) \in M.$$

So

$$f^{-1}(a, b) = f^{-1}(a, \infty) \setminus f^{-1}[b, \infty) \in M \quad \square.$$

Prop 2. Let $\Phi: \mathbb{R} \to \mathbb{R}$ be continuous.

Let $f: X \to \mathbb{R}$ be measurable.

Then $\Phi \circ f : X \to \mathbb{R}$ is measurable.

Pf. $\forall$ open G in $\mathbb{R}$, by continuity,

$$\Phi^{-1}(G) \text{ is open in } \mathbb{R}.$$

So $(\Phi \circ f)^{-1}(G) = f^{-1}\left(\Phi^{-1}(G)\right) \in M. \quad \square$

Remark: The assumption $\Phi: \mathbb{R} \to \mathbb{R}$ can be relaxed to

$\Phi: V \to \mathbb{R}$ where V is open and $V \supset \text{range}(f)$.

Prop 1.3   (i) All measurable functions on X
                    form a vector space.

        (ii) If f is measurable, then so are

$$f^2, \quad |f|, \quad f^+, \quad f^-.$$

$$\left(\; f^+_{(x)} = \begin{cases} f(x) & \text{if } f_{(x)} > 0 \\ 0 & \text{otherwise} \end{cases} \right.$$

$$\left. f^-_{(x)} = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise.} \end{cases} \right)$$

        (iii) If f, g are measurable then $f \cdot g$ is measurable.

        (iv) If f, g are measurable and $g \neq 0$ then

$$f/g \quad \text{is measurable.}$$

Pf. (i) It suffices to prove that if f, g are measurable, then so is $f + g$.

$$(f+g)^{-1}(a, \infty) = \bigcup_{\substack{s, t \in \mathbb{Q} \\ s+t > a}} \left(f^{-1}(s, \infty) \cap g^{-1}(t, \infty) \right)$$

$$\in \mathcal{M}.$$

(ii). Let $\Phi_{(x)} = x^2$. By Prop 1.2, $\Phi \circ f$ is measurable.
      But $\Phi \circ f = f^2$, and we are done.

(iii)
$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2 \right).$$

(iv) Since $g \neq 0$, range $(g) \subset \mathbb{R} \backslash \{0\}$.

Set $\Phi: \mathbb{R} \backslash \{0\} \to \mathbb{R}$ by $\Phi(x) = \frac{1}{x}$, which is cts

So by Prop 1.2, $\Phi \circ g = \frac{1}{g}$ is measurable.

Then by (iii), $\frac{f}{g} = f \cdot \frac{1}{g}$ is measurable.

◻

Prop 1.4. Let f_k, $k \geq 1$, be measurable.

Then the following are also measurable:

$$\sup_{k \geq 1} f_k \quad, \quad \inf_{k \geq 1} f_k \quad, \quad \varlimsup_{k \to \infty} f_k \quad, \quad \varliminf_{k \to \infty} f_k.$$

Pf. Let $g = \sup_{k \geq 1} f_k$.

$$g^{-1}(a, \infty) = \bigcup_{k=1}^{\infty} f_k^{-1}(a, \infty) \in \mathcal{M}.$$

$\left(g(x) > a \Leftrightarrow \exists k \text{ such that } f_k(x) > a \right).$

Hence g is measurable.

The proof for the measurability of $\inf\limits_{k} f_k$ is similar.

$$\varlimsup_{k \to \infty} f_k = \inf_{k \geq 1} \left(\sup_{j \geq k} f_j \right) \qquad \checkmark$$

$$\varliminf_{k \to \infty} f_k = \sup_{k \geq 1} \left(\inf_{j \geq k} f_j \right) \qquad \checkmark$$
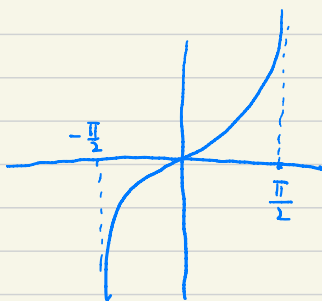
$$\blacksquare.$$

Extended real numbers.

Set $\quad \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, and call it the extended real number system.

$\overline{\mathbb{R}}$ can be viewed as the image of the function

$$\phi(x) = \tan x, \qquad x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2} \right].$$

set $\tan\left(-\tfrac{\pi}{2}\right) = -\infty, \quad \tan\left(\tfrac{\pi}{2}\right) = +\infty.$

Topological structure of $\overline{\mathbb{R}}$:

$$E \subset \overline{\mathbb{R}} \text{ is open iff } \phi^{-1}(E) \text{ is open in } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

By the above def, $E \subset \overline{\mathbb{R}}$ is open iff E is
the countable union of intervals of the form

$$[-\infty, a), \quad (a, b), \quad (a, +\infty].$$

Def: A function $f: X \to \overline{\mathbb{R}}$ is said to be measurable
if
$$f^{-1}(G) \in M \quad \text{for any open } G \text{ in } \overline{\mathbb{R}}.$$

Facts: $f: X \to \overline{\mathbb{R}}$ is measurable
iff $\quad f^{-1}(a, b), \; f^{-1}(\{+\infty\}), \; f^{-1}(\{-\infty\}) \in M.$

Prop 1.3, Prop 1.4 also hold for $\overline{\mathbb{R}}$-valued functions.

Measure spaces.

Let (X, M) be a measurable space.

Def. A measure μ on (X, M) is a function from M to $[0, +\infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) (countable additivity)

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n), \quad \text{provided}$$

$A_n \in M$ **are mutually disjoint**.

Simple facts.

- (finite additivity)

$$\mu\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} \mu(A_n) \quad \text{if} \quad A_n \in M \text{ are disjoint}.$$

- If $A \subset B$, then $\mu(B) = \mu(A) + \mu(B \setminus A)$.

$$(B = A \cup (B \setminus A))$$

Hence $\mu(A) \leq \mu(B)$.

- (Countable sub-additivity)

  Let $A_n \in M$, $n \geq 1$, then

  $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Justification: Let $B_1 = A_1$,

$$B_2 = A_2 \setminus A_1$$

$$\cdots$$

$$B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$$

$$\cdots$$

Then $B_1, B_2, \cdots$ are mutually disjoint, $B_n \in M$

$B_n \subset A_n$, moreover

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Hence $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$

$$= \sum_{n=1}^{\infty} \mu(B_n)$$

$$\leq \sum_{n=1}^{\infty} \mu(A_n).$$

Def.   The triple (X, M, μ) is called a measure space.

Prop 1.5.   Let (X, M, μ) be a measure space.

(i)   If $A_k \in M$, $k \geq 1$, is increasing in the sense

$$A_1 \subset A_2 \subset \cdots$$

then $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k).$

(ii)   If $A_k \in M$, $k \geq 1$, is decreasing in the sense

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

and assume $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k).$$

Pf.   Let us first prove (i).

Write $B_1 = A_1$

$$B_n = A_n \backslash (A_1 \cup \cdots \cup A_{n-1})$$

Then $B_1, B_2, \cdots,$ are mutually disjoint,

Moreover

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

$$\bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n \qquad (\text{check it}).$$

Hence

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= \sum_{n=1}^{\infty} \mu(B_n)$$

$$= \lim_{k \to \infty} \sum_{n=1}^{k} \mu(B_n)$$

$$= \lim_{k \to \infty} \mu\left(\bigcup_{n=1}^{k} B_n\right)$$

$$= \lim_{k \to \infty} \mu\left(\bigcup_{n=1}^{k} A_n\right)$$

$$= \lim_{k \to \infty} \mu(A_k) \qquad \left(\begin{array}{l}\text{since } A_n \\ \text{is the} \\ \text{increasing}\end{array}\right)$$

Now we prove (2).

Notice that
$$A_1 \supset A_2 \supset \cdots$$

So
$$A_1 \backslash A_1 \subset A_1 \backslash A_2 \subset \cdots \subset A_1 \backslash A_n \subset \cdots$$

Hence by (1)
$$\mu\left(\bigcup_{n=1}^{\infty} (A_1 \backslash A_n) \right) = \lim_{n \to \infty} \mu\left(A_1 \backslash A_n \right).$$

So
$$\mu\left(A_1 \backslash \left(\bigcap_{n=1}^{\infty} A_n \right) \right) = \lim_{n \to \infty} \mu(A_1 \backslash A_n). \quad \textcircled{*}$$

Since $\mu(A_1) < \infty$,

So $\mu(A_1) = \mu\left(A_1 \backslash \left(\bigcap_{n=1}^{\infty} A_n \right) \right) + \mu\left(\bigcap_{n=1}^{\infty} A_n \right)$

Hence $\mu\left(\bigcap_{n=1}^{\infty} A_n \right) = \mu(A_1) - \mu\left(A_1 \backslash \left(\bigcap_{n=1}^{\infty} A_n \right) \right)$

Similarly

$$\mu(A_n) = \mu(A_1) - \mu(A_1 \backslash A_n).$$

Plugging the above two equations into
Ⓧ, we obtain the desired
identity.

Remark: The assumption $\mu(A_1) < \infty$
can not be dropped in Prop 1.5.
Here is a counterexample:

Let $\mu = \lambda^{\frac{1}{}}$ on $\mathbb{R}$.

Let $A_n = (n, +\infty)$, $n = 1, 2, \cdots$

Then $A_n \downarrow$

However $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$

But

$$0 = \mu\left(\bigcap_{n=1}^{\infty} A_n \right) \neq \lim_{n \to \infty} \mu(A_n) = +\infty.$$

§1.4. Integration on measure spaces.

Basically we define integration in 3 steps.

    (1)   Integration of non-negative simple functions

    (2)   Integration of non-negative measurable function,

    (3) Integration of measurable functions.

Let (X, M, μ) be a measure space.

Def. (Simple functions).

    A simple function on X is a measurable real function which only take finite many values.

It has a standard form
$$S(x) = \sum_{j=1}^{N} \partial_j \, \chi_{E_j}(x),$$
where $\partial_1 < \partial_2 < \cdots < \partial_N$ and
$$E_j = \{ x : S(x) = \partial_j \} \in M.$$
$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Remark: In general, any function
$$S(x) = \sum_{j=1}^{N} \gamma_j \, \chi_{E_j}(x)$$
$$\left(\text{where } E_j \in M \right)$$
is a simple function.

Thm 1.6.   Let $f: X \to \overline{\mathbb{R}}$ be a non-negative $\overline{\mathbb{R}}$-valued measurable function. Then $\exists$ a sequence $(S_n)_{n=1}^{\infty}$ of non-negative simple functions such that

(1) $S_n \leq S_{n+1}, \quad \forall \, n \geq 1,$

(2) $f = \lim\limits_{n \to \infty} S_n.$

Pf.   Let us construct a sequence of functions $\varphi_k, \ k \geq 1,$ from $[0, \infty] \to [0, \infty)$ by

$$\varphi_k(x) = \begin{cases} \dfrac{j}{2^k} & \text{if } \ x \in [\dfrac{j}{2^k}, \dfrac{j+1}{2^k}) \\ & \qquad \text{for } j = 0, 1, \cdots, 2^k \cdot k - 1 \\ k & \text{otherwise} \end{cases}$$

Then $\quad \varphi_k(x) \uparrow x$, for any $x \geq 0$.

Take $\quad S_k^{(x)} = \varphi_k\left(f(x)\right)$, $k = 1, 2, \dots$

It is readily checked that S_k are simple,

and $\quad S_k(x) \nearrow f(x)$. $\qquad \blacksquare$.