Real Analysis

24-09-13

§1.4 Integration on measure spaces.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Let s be a simple function, i.e.

$$s = \sum_{i=1}^{N} \alpha_i \chi_{A_i},$$

with $\alpha_1 < \alpha_2 < \cdots < \alpha_N$, $A_i = \{x \in X : s(x) = \alpha_i\} \in \mathcal{M}$.

Def. Let $s = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$ (in its standard form) be a non-negative simple function. Then we define

$$\int_E s \, d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E), \quad \forall E \in \mathcal{M}.$$

Prop 1.7. Let $s = \sum_{i=1}^{N} \gamma_i \chi_{E_i}$ be a non-negative simple function (in a general form). Then

(*) $$\int_E s \, d\mu = \sum_{i=1}^{N} \gamma_i \mu(E_i \cap E), \quad \forall E \in \mathcal{M}.$$

Consequently,

$$\int_E s + t \, d\mu = \int_E s \, d\mu + \int_E t \, d\mu$$

for any other non-negative simple function t.

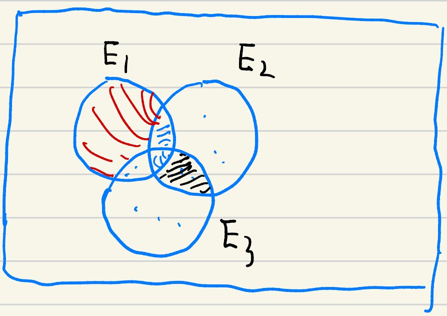
Pf. First observe that (*) holds if E_i are disjoint.

Next we prove (*) in the general case that E_i may not be disjoint. The main idea is to construct (F_j), so that F_j are disjoint, and each E_i is the union of those F_j containing in E_i

$$\left(\text{i.e.} \quad E_i = \bigcup_{j : F_j \subset E_i} F_j \right)$$

Indeed each F_j can be written as

$$A_1 \cap A_2 \cap \cdots \cap A_N \; : \qquad A_i = E_i \text{ or } E_i^c$$



Now

$$S = \sum_i \gamma_i \chi_{E_i}$$

$$= \sum_i \gamma_i \left(\sum_{j : F_j \subset E_i} \chi_{F_j} \right)$$

$$= \sum_j \left(\sum_{i:\, E_i \supset F_j} \gamma_i \right) \cdot \chi_{F_j}$$

$$= \sum_j \beta_j \chi_{F_j} \quad \left(\text{where } \beta_j = \sum_{i:\, E_i \supset F_j} \gamma_i \right)$$

Hence $\int_E s \, d\mu = \sum_j \beta_j \, \mu(F_j \cap E)$

$$= \sum_j \left(\sum_{i:\, E_i \supset F_j} \gamma_i \right) \mu(F_j \cap E)$$

$$= \sum_i \left(\sum_{j:\, F_j \subset E_i} \mu(F_j \cap E) \right) \gamma_i$$

$$= \sum_i \mu(E_i \cap E) \cdot \gamma_i \, .$$

$\square$

Next we define the integration for non-negative measurable functions.

Def: Let $f: X \to [0, +\infty]$ be measurable.

We define for $E \in \mathcal{M}$,

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f, \ s \text{ is simple} \right\}.$$

Remark: Alternatively, we can define

$$\int_E f\,d\mu = \sup\left\{ \int_E s\,d\mu : s \leq f \text{ a.e}, \ s \text{ is non-negtive simple} \right\}.$$

where $s \leq f$ a.e means $\exists\, N \in \mathcal{M}$ with $\mu(N) = 0$ so that $s \leq f$ on $X \setminus N$.

(here we use the fact if $s \leq f$ a.e, then taking $\tilde{s} = s \cdot \chi_{X\setminus N}$, then $\tilde{s} \leq f$ and $\int_E \tilde{s}\,d\mu = \int_E s\,d\mu$).

<u>Prop 1.8</u>. Let $f, g : X \to [0, +\infty]$ measurable. Then

(1) $\int_E f\,d\mu = \int_X f \cdot \chi_E\,d\mu, \ \forall\, E \in \mathcal{M}$

(2) $\int_X g\,d\mu \geq \int_X f\,d\mu$ if $g \geq f$ a.e.

Moreover, if $\int_X g\,d\mu < \infty$, then "$=$" holds iff $g = f$ a.e.

(3) $\int_{E_1} f \, d\mu \le \int_{E_2} f \, d\mu$ if $E_1 \subset E_2$

(4) $c \int_X f \, d\mu = \int_X cf \, d\mu \quad \forall \; c \ge 0.$

Pf. Here we only prove (1), i.e

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu . \quad (**).$$

We first prove that (**) holds if f is simple. To see it, let

$$s = \sum_i \alpha_i \chi_{A_i}$$

Then

$$\int_E s \, d\mu = \sum_i \alpha_i \mu(A_i \cap E)$$

$$= \int_X s \cdot \chi_E \, d\mu .$$

Next we consider the general case. We prove that

$$\int_E f \, d\mu \ge \int_X f \chi_E \, d\mu.$$

To see this, let $0 \leq s = \sum \alpha_i \chi_{A_i} \leq f \chi_E$

Then $s \cdot \chi_E = s$ and $s \leq f$.

(since $s(x)=0$ if $x \notin E$)

Hence $\int_E f \, d\mu \geq \int_E s \, d\mu$

$$= \int_X s \chi_E \, d\mu$$

$$= \int_X s \, d\mu ,$$

taking supremum of $\int_X s \, d\mu$ over $0 \leq s \leq f \chi_E$ gives

$$\int_E f \, d\mu \geq \int_X f \chi_E \, d\mu .$$

Next we prove $\int_E f \, d\mu \leq \int_X f \chi_E \, d\mu$.

To see it, let $0 \leq s \leq f$, where s is simple.

Then $s \chi_E \leq f \chi_E$, so

$$\int_X f \chi_E \, d\mu \geq \int_X s \chi_E \, d\mu = \int_E s \, d\mu ,$$

taking supremum over $0 \le s \le f$ gives

$$\int_X f \chi_E \, d\mu \ge \int_E f \, d\mu. \qquad \boxtimes$$

Prop 1.9 (Markov inequality)

Let $f: X \to [0, +\infty]$ measurable.

Let $M > 0$. Then

$$\mu\{x : f(x) \ge M\} \le \frac{1}{M} \int_X f \, d\mu.$$

Consequently (i) If $\int_X f \, d\mu < \infty$, then f is finite a.e.

(ii) If $\int_X f \, d\mu = 0$, then $f = 0$ a.e.

Pf. Write $E_M := \{x : f(x) \ge M\}$.

Then $f \ge M \cdot \chi_{E_M}$

Taking integration gives

$$\int_X f\,d\mu \geq \int_X M\,\chi_{E_M}\,d\mu = M\,\mu(E_M).$$

Hence $$\mu(E_M) \leq \frac{1}{M}\int_X f\,d\mu.$$

Next assume $\int_X f\,d\mu < \infty.$

Write

$$E_\infty = \{x : f(x) = +\infty\}.$$ Then

$$E_\infty \subset E_M \quad \forall\ M>0$$

So

$$\mu(E_\infty) \leq \mu(E_M) \leq \frac{1}{M}\int_X f\,d\mu$$

Letting $M \to +\infty$ gives $\mu(E_\infty) = 0$, i.e.

f is finite a.e.

Finally assume $\int f\,d\mu = 0.$

Let $A = \{x : f(x) > 0\}.$

Then $A = \bigcup_{n=1}^{\infty} E_{1/n}$ (clearly $A \supseteq E_{1/n}$

conversely $\forall\ x \in A$,

then $f(x) > 0$, so

$f(x) > 1/n$ for a large n

i.e $x \in E_{1/n}$ for some n)

Hence $\mu(A) \leq \sum_{n=1}^{\infty} \mu(E_{1/n})$

$$\leq \sum_{n=1}^{\infty} \frac{1}{(1/n)} \int f \, d\mu$$

$$\leq 0.$$

Hence $\mu(A) = 0$, i.e. $f = 0$ a.e $\boxed{}$.

Now suppose $f_n \to f$ a.e.

$$\left(\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e} \right),$$

Q: Do we have

$$\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu \, ?$$

Example 1. Let $\mu = \mathcal{L}_{(0,1)}$, Leb. measure on $(0,1)$.

Take $\varphi_k = 0$ on $(1/k, 1)$ and k on $(0, \frac{1}{k})$.

Then $\lim_k \varphi_k = 0$ on $(0, 1)$.

However $\int_{(0,1)} \varphi_k \, d\mu = 1$

So $\lim_k \int \varphi_k \, d\mu = 1 \neq \int \lim_k \varphi_k \, d\mu$

Example 2. Take $f_k = \chi_{[k,k+1)}$

Let $\mu = \mathcal{L}_{[0,+\infty]}$.

Again $f_k \to 0$ a.e, but

$$\lim_k \int_{(0,\infty)} f_k \, d\mu = 1 \neq \int \lim f_k \, d\mu.$$

Example 3. Take $g_k = \frac{1}{k} \chi_{[0,k]}$.

$\mu = \mathcal{L}_{[0,\infty)}$.

Thm 1.10 (Lebesgue's Monotone convergence Thm).

Let $f_k, f : X \to [0, +\infty]$ be measurable.

Assume $f_k(x) \uparrow f(x)$ on $X \setminus N$ with $\mu(N) = 0$

Then

$$\lim_{k\to\infty} \int_X f_k \, d\mu = \int_X f \, d\mu.$$

Pf. Since f_k are monotone increasing,

So are $\int_X f_k \, d\mu$.

Clearly we have $\int_X f_k \, d\mu \leq \int_X f \, d\mu$

(since $f_k \leq f$ a.e).

Hence $\lim_{k\to\infty} \int_X f_k \, d\mu \leq \int_X f \, d\mu$.

Now we prove the other direction.

Let $0 \leq s \leq f$ be simple. Let $0 < \delta < 1$.

Define: $E_k := \{ x \in X \setminus N : f_k(x) \geq \delta \cdot s(x) \}$

$k = 1, 2, \cdots$

Since $f_k(x) \uparrow f(x)$ on $X \setminus N$ and $s(x) \leq f(x)$

We have

$$\bigcup_{k=1}^{\infty} E_k = X \setminus N$$

and $E_k \subset E_{k+1}, \quad \forall k.$

Now notice that

$$f_k \geq \delta\, s(x)\, \chi_{E_k}$$

Taking integration gives

$$\int_X f_k \, d\mu \geq \delta \int s \; \chi_{E_k} \, d\mu$$

$$= \delta \int_{E_k} s \, d\mu$$

$$\left(s = \sum_{i=1}^{N} \alpha_i \chi_{A_i} \right)$$

$$= \delta \cdot \sum_{i=1}^{N} \alpha_i \, \mu(A_i \cap E_k)$$

Since $E_k \uparrow X \setminus N$, so $A_i \cap E_k \uparrow A_i \cap (X \setminus N)$

as $k \to \infty$.

Letting $k \to \infty$, we see that

$$\delta \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E_k)$$

$$\to \delta \sum_{i=1}^{N} \alpha_i \mu(A_i \cap (X \setminus N))$$

$$= \delta \sum_{i=1}^{N} \alpha_i \mu(A_i)$$

$$= \delta \cdot \int_X s \, d\mu$$

Hence

$$\lim_{k \to \infty} \int_X f_k \, d\mu \geq \delta \int_X s \, d\mu$$

Since δ is arbitrarily taken in $(0,1)$.

Letting $\delta \to 1$, $\int_X s \, d\mu \to \int_X f \, d\mu$

we have

$$\lim_{k \to \infty} \int_X f_k \, d\mu \geq \int_X f \, d\mu .$$

$\blacksquare$

Thm 1.11. (Fatou's Lemma)

Let $f_k : X \to [0,\infty]$ be measurable, $k \geq 1$.

Then

$$\int_X \varliminf_{k\to\infty} f_k \, d\mu \leq \varliminf_{k\to\infty} \int_X f_k \, d\mu .$$

Pf. Notice that

$$\varliminf_{k\to\infty} f_k(x) = \sup_{k\geq 1} \inf_{j\geq k} f_j(x)$$

Now write $g_k(x) = \inf_{j\geq k} f_j(x)$

$$g(x) = \varliminf_{k\to\infty} f_k(x).$$

Then $g_k \uparrow g$, also g_k are non-negative, measurable.

By Leb's Monotone Convergence Thm

$$\int \varliminf_{k\to\infty} f_k(x) = \int_X g \quad d\mu$$

$$= \lim_{k\to\infty} \int_X g_k \, d\mu$$

$$\leq \underline{\lim}_{k\to\infty} \int_X f_k \, d\mu \quad (\text{since } g_k \leq f_k).$$

Next we prove the linearity of integration.

<u>Prop 1.12</u>: Let $f, g : X \to [0, +\infty]$ measurable.

Let $\alpha, \beta \geq 0$.

Then
$$\int_X \alpha f + \beta g \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

Pf. First the identity holds if f, g are simple functions.

Next choose $s_k \uparrow f$, $t_k \uparrow g$, where s_k, t_k are non-negative simple.

Then $\alpha s_k + \beta t_k \nearrow \alpha f + \beta g$

So by the Monotone Convergence Thm

$$\int \alpha f + \beta g \, d\mu = \lim_{k\to\infty} \int \alpha s_k + \beta t_k \, d\mu$$

$$= \lim_{k\to\infty} \left(\alpha \int s_k \, d\mu + \beta \int t_k \, d\mu \right)$$

$$= \alpha \int f \, d\mu + \beta \int g \, d\mu .$$

$\boxtimes$.

Now we are ready to define the integration of general measurable functions.

Def. Let $f : X \to [-\infty, \infty]$ be measurable.

Then we define

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

if one of $\int_X f^+ d\mu$, $\int_X f^-$ is finite.

where $f^+ = \max\{f, 0\}$, $f^- = \max\{0, -f\}$

Def. We say a measurable function f is integrable if $\int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$.

(Notice $|f| = f^+ + f^-$. Hence by Prop 1.12,

$$\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu \,),$$

so f is integrable $\Longleftrightarrow$ $\int_X |f| d\mu < \infty$.

Prop 1.13. Let f, g be integrable and $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is integrable and

$$\int_X \alpha f + \beta g \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

Pf. We first prove $f+g$ is integrable and

$$\int f+g \, d\mu = \int f d\mu + \int g d\mu$$

Since $|f+g| \leq |f| + |g|$, so

$$\int |f+g| d\mu \leq \int |f| d\mu + \int |g| d\mu < \infty$$

Hence $f+g$ is integrable.

Now we prove $\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Notice that

$$f+g = (f+g)^+ - (f+g)^-$$
$$= (f^+ - f^-) + (g^+ - g^-)$$

Hence

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Taking integration on both sides, we obtain

$$\int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int (f+g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu$$

from which we obtain

$$\int (f+g)^+ d\mu - \int (f+g)^- d\mu = \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

i.e. $\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Next we show $c \int f \, d\mu = \int cf \, d\mu, \quad \forall c \in \mathbb{R}$.

If $c>0$, then it follows from the def of integration of meas. function since $(cf)^+ = cf^+$
$(cf)^- = cf^-$.

If $c<0$, it suffices to show

$$-\int f\,d\mu = \int -f\,d\mu.$$

Again it follows from the def. $\square$

Thm 1.14 (Lebesgue's dominated convergence Thm).

Let $f, f_k : X \to [-\infty, \infty]$ be measurable such that

$$f_k(x) \to f(x) \quad \text{a.e.} \quad \text{as } k \to \infty.$$

Moreover, suppose $\exists$ an integrable g such that

$$|f_k(x)| \le g(x) \quad \text{a.e.} \quad \text{for all } k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty} \int_X f_k\,d\mu = \int_X f\,d\mu.$$

Pf. First $|f(x)| = \lim_{k\to\infty} |f_k(x)| \le g(x)$ a.e

So f is integrable.

Now let us apply Fatou's lemma to the sequence $2g - |f_k - f|$, $k = 1, 2, \ldots$

$(\; |f_k - f| \leq |f_k| + |f| \leq 2g \quad \text{a.e})$.

We have

$$\varliminf_{k\to\infty} \int 2g - |f_k - f| \, d\mu$$
$$\geq \int \varliminf_{k\to\infty} \left(2g - |f_k - f| \, d\mu \right)$$
$$\geq \int 2g \, d\mu. \quad (***)$$

However, $\varliminf_{k\to\infty} \int 2g - |f_k - f| \, d\mu$

$$= \int 2g \, d\mu + \varliminf_{k\to\infty} (-1) \int |f_k - f| \, d\mu$$
$$= \int 2g \, d\mu - \varlimsup_{k\to\infty} \int |f_k - f| \, d\mu$$
$$\geq \int 2g \, d\mu \quad (\text{by } (***)),$$

from which we have $\varlimsup_{k\to\infty} \int |f_k - f| \, d\mu \leq 0$.

Hence $\lim_{k\to\infty} \int |f_k - f| \, d\mu = 0.$

So $\lim_{k\to\infty} \left| \int f_k \, d\mu - \int f \, d\mu \right|$

$\leq \lim_{k\to\infty} \int |f_k - f| \, d\mu = 0.$

Therefore

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu. \qquad \blacksquare$$