Stochastic Processes, Baby Queueing Theory and the Method of Stages

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Outline

1 Stochastic Processes

2 Queueing Systems

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1. Stochastic Processes

2. Queueing Systems

Stochastic Processes

- We have studied a probability system (S, Ω, P) and notion of random variable $X(w)$. Stochastic process can be defined as $X(t, w)$ where:

  $$F_{X(t)}(x) = \text{Prob}[X(t) \leq x]$$

  Example:

  1. no. of job waiting in the queue as a function of time
  2. stock market index

- Markov process

$$P[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \cdots, X(t_1) = x_1] = P[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n]$$

Continue

- Discrete-time Markov Chain
  - give example
  - $P[X_n = j \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots, X_1 = i_1] = P[X_n = j \mid X_{n-1} = i_{n-1}]$ (transition probability)
- Homogeneous Markov chain : if the transition probabilities are independent of n (or time)
- Irreducible Markov chain : if every state can be reached from every other states
- Periodic Markov chain : example : if I can reach state E_j in step γ, 2γ, 3γ, $\cdots$ where γ is > 1

Continue

- For an irreducible and aperiodic Markov Chain,we have

$$\begin{aligned}\pi_j &= \lim_{n\to\infty} \pi_j^{(n)} \\ \pi_j &= \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1\end{aligned}$$

- Example:

$$P = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

- What's the $\vec{\pi} = [\pi_0, \pi_1, \pi_2]$?

What's $\pi_j = \sum_i \pi_i P_{ij}$? (Another way to look at it)

$$\begin{aligned}
\pi_0 &= \pi_0(0) + \pi_1(\frac{1}{4}) + \pi_2(\frac{1}{4}) \\
\pi_1 &= \pi_0(\frac{3}{4}) + \pi_1(0) + \pi_2(\frac{1}{4}) \\
\pi_2 &= \pi_0(\frac{1}{4}) + \pi_1(\frac{3}{4}) + \pi_2(\frac{1}{2}) \\
& \text{The above equations are linearly dependent!} \\
1 &= \pi_0 + \pi_1 + \pi_2
\end{aligned}$$

Direct Solution:

$\pi_0 = 0.2, \pi_1 = 0.28, \pi_2 = 0.52 \Rightarrow \vec{\pi} = [0.2, 0.28, 0.52]$

Transient to limiting solution

- Define $\pi^{(n)} = [\pi_0^{(n)}, \pi_1^{(n)}, \cdots \pi_k^{(n)}]$
- Given $\pi^{(0)}$, we can perform:

$$
\begin{aligned}
\pi^{(1)} &= \pi^{(0)} P \\
\pi^{(2)} &= \pi^{(1)} P = \pi^{(0)} P^2 \\
\vdots &= \vdots \\
\pi^{(n)} &= \pi^{(0)} P^n \\
\vdots &= \vdots \\
\pi &= \pi P
\end{aligned}
$$

- look at page 33. The limiting solution (or steady state probability) is **INDEPENDENT** of the initial vector.

- For Discrete time Markov Chain
  - the number of time units that the system spends in the same state is GEOMETRICALLY DISTRIBUTED

$$(1 - P_{ii})P_{ii}^{m} \qquad \text{where m is the no. of additional steps}$$

- Homogeneous continuous time Markov chain
- $\pi Q = 0$, $\sum_i \pi_i = 1$ and $Q[i,j]$ is the rate matrix

$$\begin{aligned} q_{ij} & & = \text{rate from state i to state j} \\ q_{ii} &= -\sum_{j \neq i} q_{ij} &= \text{rate of going out of state i} \end{aligned}$$

- meaning of $\pi Q = 0$

$$\sum_j \pi_i q_{ij} = 0 \qquad \forall i$$

- Poisson Process:

$$
\begin{aligned}
P_k(t) &= \frac{(\lambda t)^k}{k!}e^{-\lambda t} \quad \text{for } k = 0, 1, 2, \ldots \\
G(Z) &= \sum_{k=0}^{\infty} P_k(t)Z^k = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}e^{-\lambda t}Z^k \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t Z)^k}{k!} = e^{-\lambda t}e^{\lambda t Z} = e^{\lambda t(Z-1)} \\
E[K] &= \frac{d}{dZ}G(Z)\mid_{Z=1} = \lambda t e^{\lambda t(Z-1)}\mid_{Z=1} = \lambda t \\
\sigma_k^2 &= \bar{K^2} - (\bar{K})^2, \text{since } \frac{d^2}{dZ^2}G(Z)\mid_{Z=1} = \bar{K^2} - \bar{K} \\
\frac{d^2}{dZ^2}G(Z) &= (\lambda t)^2 e^{\lambda t(Z-1)}\mid_{Z=1} = (\lambda t)^2 \\
\rightarrow \sigma_K^2 &= (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t
\end{aligned}
$$

- Given a Poisson, what is the distribution of it's interarrival ?

$$
\begin{aligned}
F_A(t) &= Prob[X \leq t] = 1 - P[X > t] = 1 - e^{-\lambda t} \\
f_A(t) &= \frac{dF_A(t)}{dt} = \lambda e^{-\lambda t} \quad t \geq 0 \quad \textbf{EXPONENTIAL!!}
\end{aligned}
$$

$\rightarrow$ constant rate!!

- Poisson arrival $\rightarrow$ exponential interarrival time.

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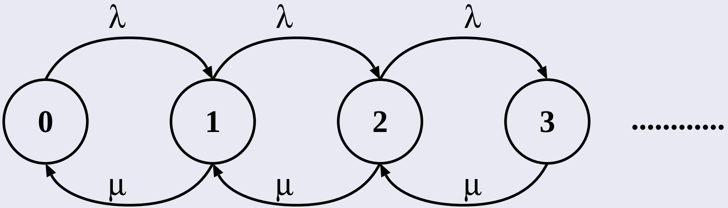
Baby Queueing Theory: $M/M/1$

Poisson arrival (or the interarrival time is exponential) and service time is exponentially distributed. Arrival is $\lambda e^{-\lambda t}$ and service is $\mu e^{-\mu t}$.



$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & \cdots \\ 0 & \mu & -(\lambda+\mu) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

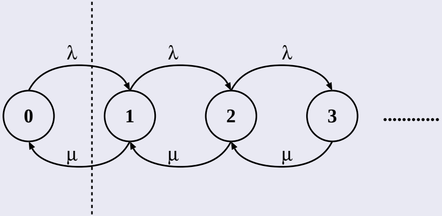
- we can use $\pi Q = 0$ and $\sum \pi_i = 1$
- For each state, flow in = flow out
- Using this, we have:

$$\begin{aligned} -\pi_0\lambda &+ \mu\pi_1 = 0 \\ \pi_0\lambda &- \pi_1(\lambda+\mu) + \mu\pi_2 = 0 \\ \pi_{i-1}\lambda &- \pi_i(\lambda+\mu) + \mu\pi_{i+1} = 0 \qquad i \geq 1 \end{aligned}$$

- Stability condition: $\frac{\lambda}{\mu} < 1$

Solution

Here, we can use flow-balance concept:



$$\pi_0\lambda = \pi_1\mu \quad \rightarrow \quad \pi_1 = \pi_0(\frac{\lambda}{\mu})$$

$$\pi_1\lambda = \pi_2\mu \quad \rightarrow \quad \pi_2 = \pi_1(\frac{\lambda}{\mu}) = \pi_0(\frac{\lambda}{\mu})^2$$

$$\pi_2\lambda = \pi_3\mu \quad \rightarrow \quad \pi_3 = \pi_2(\frac{\lambda}{\mu}) = \pi_0(\frac{\lambda}{\mu})^3$$

$$\vdots$$

In general, $\pi_i = \pi_0 (\frac{\lambda}{\mu})^i \quad i \geq 0$,

$$
\begin{aligned}
\sum_i \pi_i &= 1 \\
\pi_0 [1 + (\frac{\lambda}{\mu}) + (\frac{\lambda}{\mu})^2 + \cdots] &= 1 \\
\pi_0 [\sum_{i=0}^{\infty} (\frac{\lambda}{\mu})^i] &= 1 \\
\pi_0 [\frac{1}{1 - \frac{\lambda}{\mu}}] &= 1 \\
\pi_0 &= 1 - \frac{\lambda}{\mu}
\end{aligned}
$$

$$\pi_i = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^i \quad i \geq 0$$

($\rho = \frac{\lambda}{\mu}$ = system utilization = Prob[system or server is busy])

$$
\begin{aligned}
\bar{N} &= E[\text{number of customer in the system}] = \sum_{i=0}^{\infty} i\pi_i \\
&= \sum_{i=0}^{\infty} i(1-\rho)\rho^i = (1-\rho)\sum_{i=0}^{\infty} i\rho^i \\
&= (1-\rho)\rho\sum_{i=0}^{\infty} i\rho^{i-1} = (1-\rho)\rho\sum_{i=0}^{\infty}\frac{\partial\rho^i}{\partial\rho} \\
&= (1-\rho)\rho\frac{\partial}{\partial\rho}\sum_{i=0}^{\infty}\rho^i \\
&= (1-\rho)\rho\frac{\partial}{\partial\rho}[\frac{1}{1-\rho}] = (1-\rho)\rho[\frac{1}{(1-\rho)^2}] \\
&= \frac{\rho}{1-\rho}
\end{aligned}
$$

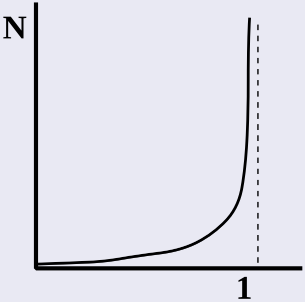
- E[number of customer waiting in the queue] = ?

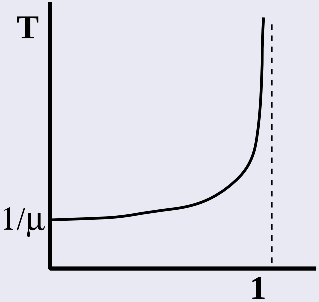
$$\begin{aligned}\sum_{k=1}^{\infty}(k-1)P_k &= \frac{\rho}{1-\rho} - \sum_{k=1}^{\infty} P_k \\ &= \frac{\rho}{1-\rho} - \rho\end{aligned}$$

$\rightarrow$ a special form, not only for $M/M/1$

Little's Result

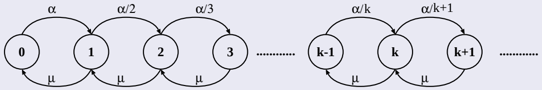
- Little's Result : $\bar{N} = \lambda \bar{T}$
  $\bar{T} = \frac{\bar{N}}{\lambda} = \frac{\frac{1}{\mu}}{1-\rho} \rightarrow$ that is why $\lambda = \mu$ is unstable





Discourage Arrivals

$$\lambda_k = \frac{\alpha}{k+1} \quad k = 0, 1, 2, \cdots \qquad \mu_k = \mu$$



$$p_0 \alpha = p_1 \mu \quad \rightarrow \quad p_1 = p_0 \frac{\alpha}{\mu}$$
$$p_1 \frac{\alpha}{2} = p_2 \mu \quad \rightarrow \quad p_2 = p_1 \frac{\alpha}{\mu} \frac{1}{2} = p_0 (\frac{\alpha}{\mu})^2 (\frac{1}{2})$$
$$p_2 \frac{\alpha}{3} = p_3 \mu \quad \rightarrow \quad p_3 = p_2 \frac{\alpha}{\mu} \frac{1}{3} = p_0 (\frac{\alpha}{\mu})^3 (\frac{1}{3})(\frac{1}{2})$$
$$\vdots$$
$$p_i = p_0 (\frac{\alpha}{\mu})^i (\frac{1}{i!}) \quad i \geq 0$$

$$\sum_{i=0}^{\infty} p_i = 1$$

$$\sum_{i=0}^{\infty} p_0 (\frac{\alpha}{\mu})^i (\frac{1}{i!}) = 1$$

$$p_0 \sum_{i=0}^{\infty} \frac{(\frac{\alpha}{\mu})^i}{i!} = 1$$

$$p_0 e^{\frac{\alpha}{\mu}} = 1$$

$$p_0 = e^{-\frac{\alpha}{\mu}}$$

$$p_i = e^{-(\frac{\alpha}{\mu})} (\frac{\alpha}{\mu})^i (\frac{1}{i!}) \quad i \geq 0$$

$$\bar{N} = ?$$

$$\bar{T} = ?$$

$$\rho \quad = \quad ? \quad \frac{\lambda}{\mu} = (1 - e^{-\frac{\alpha}{\mu}})$$

$$\begin{aligned}
\lambda \quad &= \quad \sum_{k=0}^{\infty} \frac{\alpha}{(k+1)} p_k = \sum_{k=0}^{\infty} \frac{\alpha}{k+1} \cdot \frac{e^{-\frac{\alpha}{\mu}} (\frac{\alpha}{\mu})^k}{k!} \\
&= \quad \alpha e^{-\frac{\alpha}{\mu}} \sum_{k=0}^{\infty} \frac{(\frac{\alpha}{\mu})^k}{(k+1)!} = \mu e^{-\frac{\alpha}{\mu}} \sum_{k=0}^{\infty} \frac{(\frac{\alpha}{\mu})^{k+1}}{(k+1)!} \\
&= \quad \mu e^{-\frac{\alpha}{\mu}} (e^{\frac{\alpha}{\mu}} - 1) = \mu(1 - e^{-\frac{\alpha}{\mu}})
\end{aligned}$$

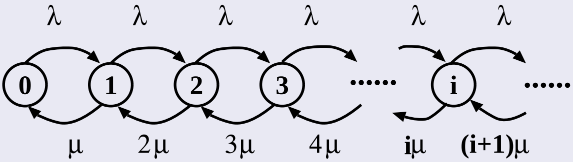
Little's Law

$\alpha(t)$ = The no. of customers arrived in $(0, t)$
$\delta(t)$ = The no. of customers departure in $(0, t)$
$N(t) = \alpha(t) - \delta(t)$ = The no. of customers in the system at time t.
$\gamma(t) = \int_0^t N(t)dt$= Total time of all entered customers have spent in the system.
λ_t = Average arrival rate $(0, t) = \frac{\alpha(t)}{t}$
T_t = System time per customer during $(0, t)$ = $\frac{\gamma(t)}{\alpha(t)}$
$\bar{N}_t$ = Average number of customer during $(0, t)$ = $\frac{\gamma(t)}{t}$
$\bar{N}_t = \frac{\gamma(t)}{t} = \frac{T_t \alpha(t)}{\frac{\alpha(t)}{\lambda_t}} = \lambda_t T_t$
Taking limit at $t \to \infty$, we have:

$$\bar{N} = \lambda \bar{T}$$

General for all algorithms e.g FIFO.....

$M/M/\infty$ system



$$P_k = P_0 \left(\frac{\lambda}{\mu}\right)^k \left(\frac{1}{k!}\right) \qquad k = 0, 1, 2, \cdots$$

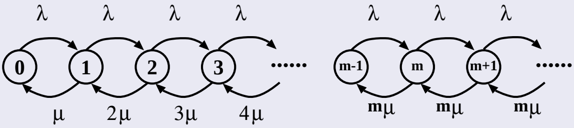
$$P_0 \left[\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \left(\frac{1}{k!}\right)\right] = 1 \qquad \Rightarrow P_0 = e^{-\lambda/\mu}$$

$$P_k = \frac{e^{-\lambda/\mu}\left(\frac{\lambda}{\mu}\right)^k}{k!} \qquad k = 0, 1, 2, \cdots$$

Continue

$$\begin{aligned}
\bar{N} &= \sum_{k=0}^{\infty} k P_k = \sum_{k=1}^{\infty} \frac{k e^{-\lambda/\mu} \left(\frac{\lambda}{\mu}\right)^k}{k!} = e^{-\lambda/\mu} \sum_{k=1}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!} \\
&= e^{-\lambda/\mu} \left(\frac{\lambda}{\mu}\right) \sum_{k=1}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^{k-1}}{(k-1)!} = \frac{\lambda}{\mu} \\
\bar{T} &= \bar{N}/\lambda = \frac{1}{\mu}
\end{aligned}$$

$M/M/m$ system



$\lambda_k = \lambda$ for $k = 0, 1, \cdots$.
$\mu_k = k\mu$ for $0 \leq k \leq m$ and $m\mu$ for $k \geq m$.

$$p_0\lambda = p_1\mu \Rightarrow p_1 = p_0\left(\frac{\lambda}{\mu}\right)$$

$$p_1\lambda = p_2 2\mu \Rightarrow p_2 = p_0\left(\frac{\lambda}{\mu}\right)^2\left(\frac{1}{2}\right)$$

$$\vdots$$

$$p_k = p_0\left(\frac{\lambda}{\mu}\right)^k\left(\frac{1}{k!}\right) \quad k = 0, 1, \ldots, m$$

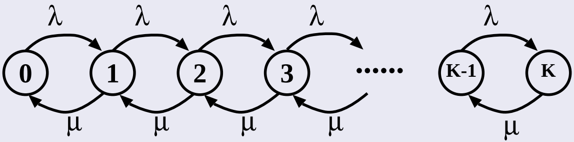
continue

for $k \geq m$

$$
\begin{aligned}
p_m\lambda &= p_{m+1}m\mu \quad \Rightarrow \quad p_{m+1} = p_0\left(\frac{\lambda}{\mu}\right)^{m+1}\left(\frac{1}{m!}\right)\left(\frac{1}{m}\right)\\
p_{m+1}\lambda &= p_{m+2}m\mu \quad \Rightarrow \quad p_{m+2} = p_0\left(\frac{\lambda}{\mu}\right)^{m+2}\left(\frac{1}{m!}\right)\left(\frac{1}{m}\right)^2\\
&\vdots\\
p_k &= p_0\left(\frac{\lambda}{\mu}\right)^{k}\left(\frac{1}{m!}\right)\left(\frac{1}{m}\right)^{(k-m)} \qquad k \geq m
\end{aligned}
$$

$$
\text{Prob[queueing]} = \sum_{k=m}^{\infty} p_k
$$

$M/M/1/K$ finite storage system

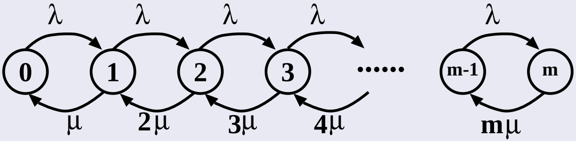


$$p_k = p_0 \left(\frac{\lambda}{\mu}\right)^k \qquad k = 0, 1, \ldots, K$$

$$p_0 = \left[\sum_{k=0}^{K} \left(\frac{\lambda}{\mu}\right)^k\right]^{-1} = \left[\frac{1-\rho^{K+1}}{1-\rho}\right]^{-1} = \frac{1-\rho}{1-\rho^{K+1}} \quad \text{where } \rho = \frac{\lambda}{\mu}$$

Prob[blocking] $=$?
Using the similar approach, we can find $\bar{N}$ and $\bar{T}$.
Average arrival rate is $\lambda(1 - P_K)$.

$M/M/m/m$ ($m-$server loss system)



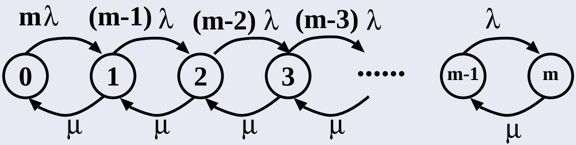
$\lambda_k = \lambda$ for $k < m$ and zero otherwise. $\mu_k = k\mu$ for $k = 1, 2, \ldots, m$.

$$p_k = p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \quad k \leq m$$

$$p_0 = \left[\sum_{k=0}^{m} \frac{(\lambda/\mu)^k}{k!}\right]^{-1}$$

Prob[all servers are busy] is equal to P_m.

$M/M/1//m$ (Finite customer population)



$$
\begin{aligned}
p_k &= p_0 \Pi_{i=0}^{k-1} \frac{\lambda(M-i)}{\mu} \qquad 0 \leq k \leq M \\
&= p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!} \qquad 0 \leq k \leq M \\
p_0 &= \left[\sum_{i=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}\right]^{-1}
\end{aligned}
$$

p_k to r_k

- What is p_k? It is $\lim_{t\to\infty} \frac{\text{sum of time slots with } k \text{ customers}}{t}$
- r_k = Prob[arriving customer finds the system in state E_k]
- Is $p_k = r_k$?
- Let us look at $D/D/1$, interarrival time is 4 secs, service time is 3 sec. Then $p_0 = 1/4$ and $r_0 = 1$.

More

- For Poisson arrival, $P_k(t) = R_k(t)$ or $p_k = r_k$.

$$\begin{aligned} R_k(t) &= \lim_{\delta t \to 0} P[N(t) = k | A(t+\delta t)] = \frac{P[N(t) = k, A(t+\delta t)]}{P[A(t+\delta t)]} \\ &= \frac{P[A(t+\delta t) | N(t) = k] P[N(t) = k]}{P[A(t+\delta t)]} \end{aligned}$$

- Due to memoryless property:

$$P[A(t+\delta t) | N(t) = k] = P[A(t+\delta t)]$$

- Due to independence,

$$R_k(t) = P[N(t) = k] = p_k(t)$$

Method of stages: Erlangian distribution E_r

Let service time density function

$$b(x) = \mu e^{-\mu x} \qquad x \geq 0$$

$$B^*(s) = \frac{\mu}{s+\mu}; E[\tilde{x}] = \frac{1}{\mu}; \sigma_b^2 = \frac{1}{\mu^2}$$

$$h(y) = 2\mu e^{-2\mu y} \qquad y \geq 0$$

$$x = y + y$$

$$B^*(s) = \left(\frac{2\mu}{s+2\mu}\right)^2$$

continue

From (2.146)

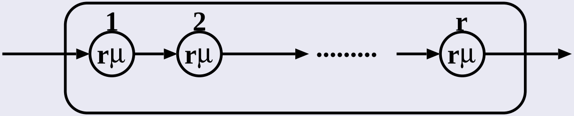
$$X^*(s) = \left(\frac{\lambda}{s+\lambda}\right)^k \Rightarrow f_X(x) = \frac{\lambda(\lambda x)^{k-1}}{(k-1)!}e^{-\lambda x} \quad x \geq 0; k \geq 1$$

Therefore:

$$b(x) = 2\mu(2\mu x)e^{-2\mu x} \quad x \geq 0$$

$$E[x] = E[y] + E[y] = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu}; \quad \sigma_b^2 = \sigma_n^2 + \sigma_n^2 = \frac{2}{(2\mu)^2} = \frac{1}{2\mu^2}$$

E_r: $r-$stage Erlangian distribution



$$
\begin{aligned}
h(y) &= r\mu e^{-r\mu y} \qquad y \geq 0 \\
E[y] &= \frac{1}{r\mu} \quad ; \quad \sigma_h^2 = \frac{1}{(r\mu)^2} \quad ; \quad E[x] = r\frac{1}{r\mu} = \frac{1}{\mu} \\
\sigma_x^2 &= r\left(\frac{1}{r\mu}\right)^2 = \frac{1}{r\mu^2} \\
B^*(s) &= \left[\frac{r\mu}{s+r\mu}\right]^r \Rightarrow b(x) = \frac{r\mu(r\mu x)^{r-1}}{(r-1)!}e^{-r\mu x} \qquad x \geq 0
\end{aligned}
$$

$M/E_r/1$ system

$$
\begin{aligned}
a(t) &= \lambda e^{-\lambda t} \\
b(t) &= \frac{r\mu(r\mu x)^{r-1}}{(r-1)!} e^{(r\mu x)} \quad x \geq 0
\end{aligned}
$$

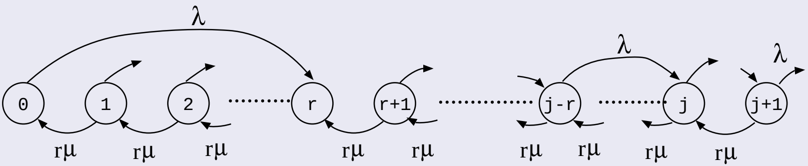
State description: $[k, s_l]$ transform to $[s]$ where s is the total number of stages yet to be completed by all customers.
If the system has k customers and when the i^{th} stage of service contains the customers.

$$
\begin{aligned}
j &= \text{number of stages left in the total system} \\
&= (k-1)r + (r-i+1) = rk - i + 1
\end{aligned}
$$

$M/E_r/1$ system: continue

Let P_j be the probability of j stages of work in the system. Since $j = rk - i + 1$, we have:

$$p_k = \text{Prob}[k \text{ customers}] = \sum_{j=(k-1)r+1}^{j=rk} P_j$$



Let $P_j = 0$ for $j < 0$.

$$\begin{aligned} \lambda P_0 &= r\mu P_1 \\ (\lambda + r\mu)P_j &= \lambda P_{j-r} + r\mu P_{j+1} \end{aligned}$$

Define $P(Z) = \sum_{j=0}^{\infty} P_j Z^j$

$$\begin{aligned} \sum_{j=1}^{\infty}(\lambda + r\mu)P_j Z^j &= \sum_{j=1}^{\infty} \lambda P_{j-r} Z^j + \sum_{j=1}^{\infty} r\mu P_{j+1} Z^j \\ (\lambda + r\mu)\left[P(Z) - P_0\right] &= \lambda Z^r \left[P(Z)\right] + \frac{r\mu}{Z}\left[P(Z) - P_0 - P_1 Z\right] \end{aligned}$$

$$\begin{aligned} P(Z) &= \frac{P_0\left[\lambda + r\mu - (r\mu/Z)\right] + r\mu P_1}{\lambda + r\mu - \lambda Z^r - (r\mu/Z)} \\ &= \frac{P_0 r\mu(1 - 1/Z)}{\lambda + r\mu - \lambda Z^r - (r\mu/Z)} \end{aligned}$$

Since $P(1) = 1$, therefore, using L' Hospital rule, we have $1 = \frac{r\mu P_0}{r\mu - \lambda r}$, therefore, $P_0 = \frac{r\mu - \lambda r}{r\mu} = 1 - \frac{\lambda}{\mu}$.
Define $\rho = \frac{\lambda}{\mu}$, we have

$$P(Z) = \frac{r\mu(1-\rho)(1-Z)}{r\mu + \lambda Z^{r+1} - (\lambda + r\mu)Z}$$

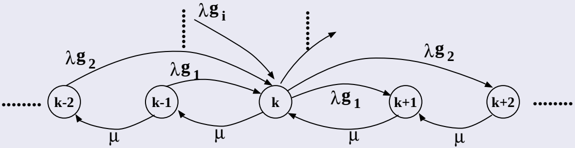
For general r, look at denominator, there are $(r+1)$ zeros. Unity is one of them; we have $(1-Z)[r\mu - \lambda(Z + Z^2 + \cdots + Z^r)]$. Therefore, we have r zeros which are $Z_1, Z_2, \cdots, Z_r$. We can arrange them to be $r\mu(1 - Z/Z_1)(1 - Z/Z_2)\cdots(1 - Z/Z_r)$. We have:

$$P(Z) = (1-\rho)\sum_{i=1}^{r} \frac{A_i}{1 - Z/Z_i}$$

Need to resolve this by partial fraction expansion.
For $E_r/M/1$ system, derive it at home.

Bulk Arrival System

Let g_i be the probability that the bulk size is i, for $i > 0$.



$$
\begin{aligned}
\lambda P_0 &= \mu P_1 \\
(\lambda+\mu)P_k &= \mu P_{k+1} + \sum_{i=0}^{k-1} \lambda g_{k-i} P_i \\
(\lambda+\mu)\sum_{k=1}^{\infty} P_k Z^k &= \mu \sum_{k=1}^{\infty} P_{k+1} Z^k + \lambda \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} g_{k-i} P_i Z^K \\
(\lambda+\mu)\left[P(Z)-P_0\right] &= \frac{\mu}{Z}\left[P(Z)-P_0-P_1 Z\right] + \lambda P(Z)G(Z)
\end{aligned}
$$

Analysis: continue

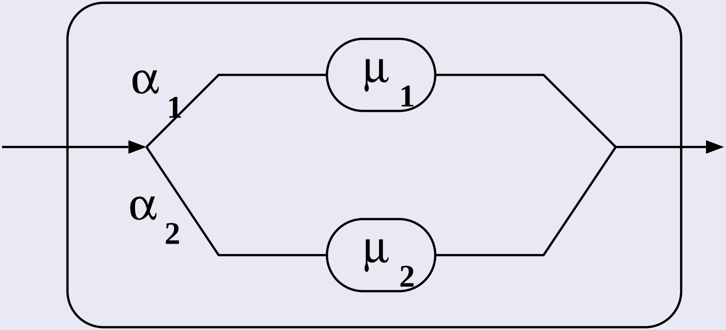
$$P(Z) = \frac{\mu P_0(1 - Z)}{\mu(1 - Z) - \lambda Z[1 - G(Z)]}$$

Using $P(1) = 1$ and L' Hospital rule

$$P(Z) = \frac{\mu(1 - \rho)(1 - Z)}{\mu(1 - Z) - \lambda Z[1 - G(Z)]}$$

where $\rho = \frac{\lambda G'(1)}{\mu}$
For bulk service system, try it at home.

Parallel System



$$b(x) = \alpha_1 \mu_1 e^{-\mu_1 x} + \alpha_2 \mu_2 e^{-\mu_2 x} \quad x \geq 0$$

$$B^*(s) = \alpha_1 \left(\frac{\mu_1}{s + \mu_1} \right) + \alpha_2 \left(\frac{\mu_2}{s + \mu_2} \right)$$

Continue:

In general, if we have R parallel stages (hyper-exponential):

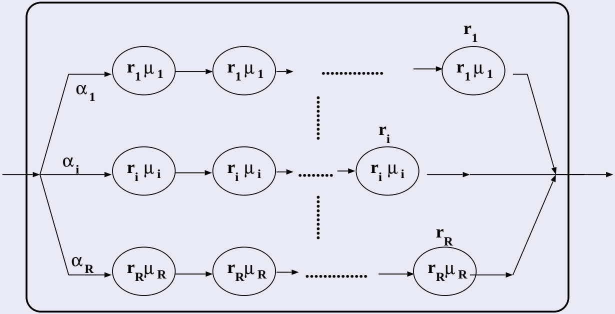
$$B^*(s) = \sum_{i=1}^{R} \alpha_i \left(\frac{\mu_i}{s + \mu_i} \right)$$

$$b(x) = \sum_{i=1}^{R} \alpha_i \mu_i e^{-\mu_i x} \quad x \geq 0$$

$$\bar{x} = \sum_{i=1}^{R} \alpha_i \left(\frac{1}{\mu_i} \right) \qquad \overline{x^2} = \sum_{i=1}^{R} \alpha_i \left(\frac{2}{\mu_i^2} \right)$$

$$C_b^2 = \frac{\sigma_b^2}{(\bar{x})^2} = \frac{\overline{x^2} - (\bar{x})^2}{(\bar{x})^2} \quad \Rightarrow \quad C_b^2 \geq 1$$

Series and Parallel System



$$b(x) \;=\; \sum_{i=1}^{R} \alpha_i \frac{r_i\mu_i(r_i\mu_i x)^{r_i-1}}{(r_i-1)!} e^{-r_i\mu_i x} \qquad x \geq 0$$

$$B^*(s) \;=\; \sum_{i=1}^{R} \alpha_i \left(\frac{r_i\mu_i}{s + r_i\mu_i}\right)^{r_i}$$