$G/G/1$ Queueing Systems

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Outline

1. Lindley's Integral Equation
2. Spectral Solution to Lindley's Integral Equation
3. Example

Outline

Notations

- $A(t)$ PDF for interarrival times between customers.
- $B(x)$ PDF of service time for customers (independent).
- Service discipline: FCFS
- C_n, the n^{th} arriving customer
- $t_n = \tau_n - \tau_{n-1}$, interarrival time between C_n and C_{n-1}
- x_n, service time of C_n
- w_n, waiting time (in queue) for C_n

Description

Random variables $\{t_n\}$ and $\{x_n\}$ are independent and described by $A(t)$ and $B(x)$ independent of index n.

Working equation

- The waiting times for C_{n+1} is

$$w_{n+1} = \begin{cases} w_n + x_n - t_{n+1} & \text{if } w_n + x_n - t_{n+1} \geq 0, \\ 0 & \text{if } w_n + x_n - t_{n+1} < 0. \end{cases} \quad (1)$$

- Define the random variable:

$$u_n = x_n - t_{n+1}.$$

For stable system, we require $\lim_{n\to\infty} E[u_n] < 0$, or

$$\lim_{n\to\infty} E[u_n] = \lim_{n\to\infty} \{E[x_n] - E[t_{n+1}]\} = \bar{x} - \bar{t} = \bar{t}(\rho - 1)$$

- Combining the above equations, we have

$$w_{n+1} = \begin{cases} w_n + u_n & \text{if } w_n + u_n \geq 0, \\ 0 & \text{if } w_n + u_n < 0. \end{cases} \quad (2)$$

Continue

- We can write down w_n as:

$$w_{n+1} = \max[0, w_n + u_n] = (w_n + u_n)^+ \tag{3}$$

- We aim to derive $\lim_{n\to\infty} P[w_n \le y] = W(y)$, which exists when $\rho < 1$.
- To derive $W(y)$, we first define $C_n(u)$ as the PDF for u_n,

$$C_n(u) = P[u_n = x_n - t_{n+1} \le u]$$

- Derivation of $C_n(u)$:

$$\begin{aligned} C_n(u) &= P[x_n - t_{n+1} \le u] = \int_{t=0}^{\infty} P[x_n \le u + t | t_{n+1} = t] dA(t) \\ &= \int_{t=0}^{\infty} B(u+t) dA(t) = C(u) \end{aligned}$$

Continue

- For $W(y)$, when $y \geq 0$, we have

$$W_{n+1}(y) = P[w_n + u_n \leq y] = \int_{w=0^-}^{\infty} P[u_n \leq y - w | w_n = w] dW_n(w)$$

- Since u_n is independent of w_n, we have

$$W_{n+1}(y) = \int_{w=0^-}^{\infty} C_n(y - w) dW_n(w) \quad \text{for } y \geq 0$$

- Taking $\lim_{n\to\infty}$, we have the Lindley's Integral Equation:

$$W(y) = \int_{w=0^-}^{\infty} C(y - w) dW(w) \quad \text{for } y \geq 0 \tag{4}$$

Further, it is clear that

$$W(y) = 0 \quad \text{for } y < 0 \tag{5}$$

Second form of Lindley's Equation

- For the previous Lindley's equation, we do integration by part:

$$
\begin{aligned}
W(y) &= C(y-w)W(w)\Big|_{w=0^-}^{\infty} - \int_{0^-}^{\infty} W(w)dC(y-w) \\
&= \lim_{w\rightarrow\infty} C(y-w)W(w) - C(y)W(0^-) - \int_{0^-}^{\infty} W(w)dC(y-w)
\end{aligned}
$$

- Since $C(y-w)=0$ as $w \rightarrow \infty$ since it corresponds to the probability that an interarrival time approaches infinity, so the probability goes to zero if $A(t)$ has to have a finite moment. Similarly $w(0^-)=0$ so we have

$$
W(y) = \begin{cases} -\int_{w=0^-}^{\infty} W(w)dC(y-w) & y \geq 0 \\ 0 & y < 0. \end{cases} \quad (6)
$$

Third form of Lindley's Equation

- For the third form,consider the simple variable change $u = y - w$ for the argument of our distributions, we have

$$W(y) = \begin{cases} \int_{u=-\infty}^{y} W(y-u)dC(u) & y \geq 0, \\ 0 & y < 0. \end{cases} \tag{7}$$

- The remaining issue is, how to solve the Lindley's equation for $G/G/1$ queue ?

Outline

Spectral Solution

- If we examine Lindley's Equation in (7), it is "almost" like a convolution except on the half plane.
- Define a "complementary" waiting time

$$W_{-}(y) = \begin{cases} 0 & y \geq 0, \\ \int_{u=-\infty}^{y} W(y-u)dC(u) & y < 0. \end{cases} \tag{8}$$

- Adding Equation (7) and (8):

$$W(y) + W_{-}(y) = \int_{-\infty}^{y} W(y-u)c(u)du \quad \text{for all real } y \tag{9}$$

where we denote the pdf for $\tilde{u}$ by $c(u) = dC(u)/du$.

Spectral Solution (continue)

- We now need to define transform for our various functions. Laplace transform of $W_-(y)$ is

$$\Phi_-(s) = \int_{-\infty}^{\infty} W_-(y)e^{-sy}dy = \int_{-\infty}^{0} W_-(y)e^{-sy}dy. \tag{10}$$

- Laplace transform for our waiting time PDF $W(y)$,

$$\Phi_+(s) = \int_{-\infty}^{\infty} W(y)e^{-sy}dy = \int_{0^-}^{\infty} W(y)e^{-sy}dy. \tag{11}$$

- Note that $\Phi_+(s)$ is the Laplace transform of the PDF for waiting time, our usual definition of transform is in terms of pdf. Let $W^*(s)$ be the transform for the waiting time. Therefore, we have:

$$s\Phi_+(s) = W^*(s). \tag{12}$$

Spectral Solution (continue)

- Since we define the pdf of $\tilde{u}$ as $c(u) = dC(u)/du = a(-u) \otimes b(u)$, we have

$$C^*(s) = A^*(-s)B^*(s)$$

- Taking the Laplace transform of Eq. (9), we have

$$\Phi_+(s) + \Phi_-(s) = \Phi_+(s)C^*(s) = \Phi_+(s)A^*(-s)B^*(s)$$

This gives us

$$\Phi_-(s) = \Phi_+(s)\left[A^*(-s)B^*(s) - 1\right] \tag{13}$$

- We consider queueing systems for which $A^*(s)$ and $B^*(s)$ are *rational functions* of s, where

$$A^*(-s)B^*(s) - 1 = \frac{\Psi_+(s)}{\Psi_-(s)} \tag{14}$$

Spectral Solution (continue)

- Putting Eq (14) into (13), we have

$$\Phi_-(s) = \Phi_+(s)\frac{\Psi_+(s)}{\Psi_-(s)}$$

- Applying the Liouville's theorem (e.g., if $f(z)$ is analytic and bounded for all finite values of z, then $f(z)$ is a constant), we have

$$\Phi_-(s)\Psi_-(s) = \Phi_+(s)\Psi_+(s) = K.$$

or

$$\Phi_+(s) = \frac{K}{\Psi_+(s)} \tag{15}$$

- We need to determine K. Note that

$$s\Phi_+(s) = W^*(s) = \int_{y=0^-}^{\infty} e^{-sy}\,dW(y)$$

Spectral Solution (continue)

- Taking the limit of $s \rightarrow 0$, we have:

$$\lim_{s \rightarrow 0} s\Phi_+(s) = \lim_{s \rightarrow 0} \int_{0^-}^{\infty} e^{-sy} dW(y) = 1 = \lim_{s \rightarrow 0} \frac{sK}{\Psi_+(s)}$$

or

$$K = \lim_{s \rightarrow 0} \frac{\Psi_+(s)}{s} \tag{16}$$

Summary

To analyze $G/G/1$, do the following.

1. Use $A^*(-s)B^*(s) - 1 = \frac{\Psi_+(s)}{\Psi_-(s)}$ to determine $\Psi_+(s)$ and $\Psi_-(s)$.
2. Use $K = \lim_{s \to 0} \frac{\Psi_+(s)}{s}$ to determine K.
3. Use $\Phi_+(s) = \frac{K}{\Psi_+(s)}$ to determine the functional form of $\Phi_+(s)$.
4. Given $\Phi_+(s)$, we use $s\Phi_+(s) = W^*(s)$ to find the Laplace transform for the pdf for $\tilde{w}$.

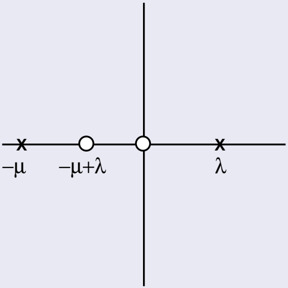
Outline

$M/M/1$

- We have $A^*(s) = \lambda/(s+\lambda)$ and $B^*(s) = \mu/(s+\mu)$. Then

$$A^*(-s)B^*(s) - 1 = \left(\frac{\lambda}{\lambda - s}\right)\left(\frac{\mu}{s+\mu}\right) - 1 = \frac{s^2 + s(\mu-\lambda)}{(\lambda - s)(s+\mu)}$$

- We have $\frac{\Psi_+(s)}{\Psi_-(s)} = \frac{s^2+s(\mu-\lambda)}{(\lambda-s)(s+\mu)} = \frac{s(s+\mu-\lambda)}{(s+\mu)(\lambda-s)}$
- Examining the poles and zeros, we have:



$M/M/1$ continue

- Note that $\Psi_+(s)$ must be analytic and zero-free for $Re(s) > 0$. Collecting the two zeros and one pole for $\Psi_+(s)$, We have:

$$\Psi_+(s) = \frac{s(s+\mu-\lambda)}{s+u} \quad (17)$$
$$\Psi_-(s) = \lambda - s \quad (18)$$

- Determine K:

$$K = \lim_{s\to 0} \frac{\Psi_+(s)}{s} = \lim_{s\to 0} \frac{s+\mu-\lambda}{s+\mu} = 1-\rho.$$

- We have $\Phi_+(s) = \frac{(1-\rho)(s+\mu)}{s(s+\mu-\lambda)}$.
- Since $W^*(s) = s\Phi_+(s)$, we can invert $W^*(s)$, which is

$$W(y) = 1 - \rho e^{\mu(1-\rho)y} \quad y \geq 0.$$

Summary

- In general, analyzing $G/G/1$ involves complex analysis, as well as transform inversion.
- These two procedures are complicated and involved in general.
- In many cases, we need numerical methods to perform the transform inversion.
- We can consider other means to provide *approximation* to $G/G/1$ analysis.