THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2023-2024 Term 2
Homework Assignment 7
Due Date: 28 April, 2024 (Sunday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

______________________________ ______________________________
Signature Date

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2023R2 Introductory Probability (MATH3280B)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

  For the declaration sheet:

  Either

Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence “I have read the university regulations.”

- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit a digital form via Gradescope. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Please be aware that you can only use a ball-point pen to write your answers for any exams.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Show that X and Y are identically distributed and not necessarily independent, then

$$\mathrm{Cov}(X+Y, X-Y) = 0.$$

2. Let X have moment generating function $M(t)$, and define $\Phi(t) = \log M(t)$. Show that

$$\Phi''(t)\big|_{t=0} = \mathrm{Var}(X).$$

3. Suppose that $X$is a random variable with mean and variance both equal to 20. What can be said about $P\{0 < X < 40\}$?

4. Let $X_1, \cdots, X_{20}$ be independent Poisson random variables with mean 1.

   (a) Use the Markov inequality to obtain a bound on

   $$P\left\{\sum_1^{20} X_i > 15\right\}.$$

   (b) Use the central limit theorem to approximate

   $$P\left\{\sum_1^{20} X_i > 15\right\}.$$

5. Let Z_n, $n \geq 1$, be a sequence of random variables and c a constant such that, for each

   $$\epsilon > 0,\ P\{|Z_n - c| > \epsilon\} \to 0 \qquad \text{as} \qquad n \to \infty.$$

   Show that, for any bounded continuous function g,

   $$E[g(Z_n)] \to g(c) \qquad \text{as} \qquad n \to \infty.$$

6. Let $f(x)$ be a continuous function defined for $0 \leq x \leq 1$ Consider the functions

   $$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

   (called Bernstein polynomials) and prove that

   $$\lim_{n\to\infty} B_n(x) = f(x).$$

   Hint: Let $X_1, X_2, \cdots$ be independent Bernoulli random variables with mean x. Show that

   $$B_n(x) = E\left[f\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)\right]$$

   and then use the result of Problem 5.

   Since it can be shown that the convergence of $B_n(x)$ to $f(x)$ is uniform in x, the preceding reasoning provides a probabilistic proof of the famous Weierstrass theorem of analysis, which states that any continuous function on a closed interval can be approximated arbitrarily closely by a polynomial.

7. You may answer one of the following problems below, but not both.

- If X is a Poisson random variable with mean λ, show that for $i > \lambda$,

$$P\{X \geq i\} \leq \frac{e^{-\lambda}(e\lambda)^i}{i^i}.$$

- If X is a Poisson random variable with mean λ, show that for $i < \lambda$,

$$P\{X \leq i\} \leq \frac{e^{-\lambda}(e\lambda)^i}{i^i}.$$