MATH 3270 A : Ordinary Differential Equations

Midterm, October, 2015

Suggested Solutions.

1. (12 pts)

(a) Let $M = \frac{\sin y}{y} - 2e^{-x}\sin x$.

$N = \frac{\cos y + 2e^{-x}\cos x}{y}$

$M_y = \frac{y\cos y - \sin y}{y^2}$, $N_x = \frac{-2e^{-x}\cos x - 2e^{-x}\sin x}{y}$,

$M_y \neq N_x$, so it's not exact. 2'

(b) Let $\mu(x,y)$ be the integrating factor.

After multiplying μ to the equation,

$$\left.\begin{aligned}(\mu M)_y &= (\mu N)_x \\ \mu_y M - \mu_x N &= \mu(N_x - M_y)\end{aligned}\right\} 2'$$

Substitute M, N, M_y, N_x,

$\sin y(y\mu_y - \mu) - 2e^{-x} y \sin x (y\mu_y - \mu)$

$-2ye^{x}\cos x(\mu_x - \mu) - y\cos y(\mu_x - \mu) = 0$

Let $y\mu_y - \mu = 0$, $\mu_x - \mu = 0$. 2'

WLOG, choose $\mu(x,y) = ye^x$. 2'

(c) Multiply $\mu = ye^x$ to the equation.

$$(e^x \sin y - 2y\sin x)dx + (e^x\cos y + 2\cos x)dy = 0$$

Suppose $F(x,y) = C$ is the implicit solution,

$$\begin{cases} F_x = e^x \sin y - 2y\sin x & (1) \\ F_y = e^x\cos y + 2\cos x & (2) \end{cases}$$ 2'

Solve (1) $F(x,y) = e^x \sin y + 2y\cos x + f(y)$.

differentiate F with respect to y.

$$F_y = e^x\cos y + 2\cos x + f'(y)$$

compare to (2), $f'(y) = 0 \Rightarrow f(y) = C$.

Hence the solution is

$$e^x \sin y + 2y\cos x = C$$ 2'

where C is arbitrary constant.

2.(a) (10 pts) Solve the homogeneous equation.

$$y'' + 2y' + 6y = 0.$$

characteristic equation:

$$r^2 + 2r + 6 = 0, \quad 2'$$

roots are, $r_1 = -1 + \sqrt{5}i$, $r_2 = -1 - \sqrt{5}i$; $1'$

General solution to homogeneous one:

$$y_h(t) = e^{-t}\left(C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}t)\right), \quad 2'$$

C_1, C_2 constants.

Suppose $Y(t) = e^{-t}(A\sin 2t + B\cos 2t)$ is a particular solution, $2'$

with A, B to be determined.

$$Y'(t) = e^{-t}\left[(-A-2B)\sin 2t + (2A-B)\cos 2t\right]$$

$$Y''(t) = e^{-t}\left[(-3A+4B)\sin 2t + (-4A-3B)\cos 2t\right]$$

$Y'' + 2Y' + 6Y = 4e^{-t}\sin 2t$ implies,

$$A = 4, \quad B = 0. \quad 1'$$

so the particular solution is $Y(t) = 4e^{-t}\sin 2t$.

General solution to the inhomogeneous equation:

$$y(t) = y_h(t) + Y(t) = e^{-t}(C_1\cos\sqrt{5}t + C_2\sin\sqrt{5}t) + 4e^{-t}\sin 2t.$$

Substitute initial conditions; $\begin{cases} C_1 = 1 \\ -C_1 + \sqrt{5}C_2 + 8 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -\frac{6}{\sqrt{5}}. \end{cases}$ $2'$

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(b) (10 pts) Solve homogeneous equation:

$$y''' + y' = 0.$$

Characteristic equation:

2 $r^3 + r = 0. \Rightarrow r_1 = 0.\ r_2 = i,\ r_3 = -i.$

General solution to the homogeneous equation.

1 $y_h(t) = C_1 + C_2 \cos t + C_3 \sin t.$

Suppose $Y(t)$ ~~is~~ a particular solution.

$$(Y'' + Y)' = \sec t = (\ln|\sec t + \tan t| + C)'$$

$$\Rightarrow Y'' + Y = \ln|\sec t + \tan t| + C$$

WLOG, let $C = 0$.

$$y_1(t) = \sin t,\ y_2(t) = \cos t,\ g(t) = \ln|\sec t + \tan t|$$

$$W[y_1, y_2] = -\sin^2 t - \cos^2 t = -1$$

particular solution,

4 $$Y_p(t) = -y_1(t)\int \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2(t)\int \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds$$

$$= \ln|\sec t + \tan t| + \sin t \ln|\cos t| - t\cos t$$

General solution to the inhomogeneous equation.

$$y(t) = y_h(t) + Y_p(t) = C_1 + C_2\cos t + C_3 \sin t + \ln|\sec t + \tan t| + \sin t \ln|\cos t| - t\cos t.$$

3 Substitute initial conditions: $C_1 = 0,\ C_2 = 2,\ C_3 = 2$

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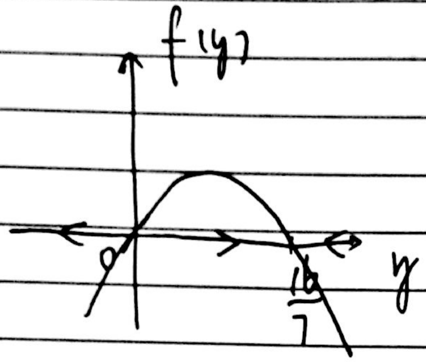
(10 pts)

3. (a) $f(y) = \frac{1}{3}\left(\frac{4}{7} - \frac{y}{4}\right)y$

$f(y) = 0 \Rightarrow y_1 = 0,\ \ y_2 = \frac{16}{7}$

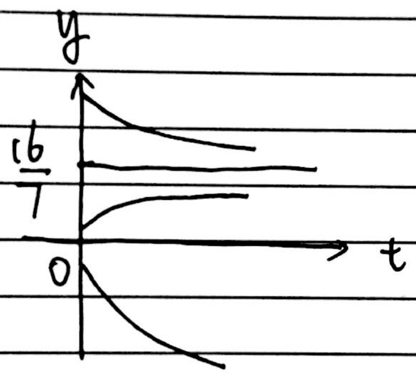
(b)



$y_1 = 0$, unstable

$y_2 = \frac{16}{7}$, asymptotically stable

(c)



(12 pts)

4. (a) $W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} t^2+t & t^2-1 \\ 2t+1 & 2t \end{vmatrix}$

$= (t+1)^2$

(b) Yes. Since $W[y_1, y_2](0) = 1 \neq 0$, $0 \in I = (-2, 1)$

(c) No. By Abel's formula,

$$W[y_1, y_2](t) = c\exp\left(-\int p(t)\,dt\right)$$

If $W = 0$ for some $t_0 \in I$, then $W \equiv 0$ on I.

Proof: $W = y_1 y_2' - y_1' y_2$.

$y_1'' + p y_1' + q y_1 = 0$

$y_2'' + p y_2' + q y_2 = 0$

$$\begin{aligned}
W' &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\
&= y_1 y_2'' - y_1'' y_2 \\
&= y_1(-p y_2' - q y_2) - (-p y_1' - q y_1) y_2 \\
&= -p y_1 y_2' - q y_1 y_2 + p y_1' y_2 + q y_1 y_2 \\
&= -p(y_1 y_2' - y_1' y_2) \\
&= -pW
\end{aligned}$$

$\Rightarrow W = c\exp\left(\int p(t)\,dt\right)$

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5. (12 pts)

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = x\frac{dy}{dx} = xy'$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt}$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

$$= xy' + x^2y'' = \frac{dy}{dt} + x^2y''$$

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}\right)$$

$$= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt} + 2x\frac{dx}{dt}\frac{d^2y}{dx^2} + x^2\frac{d^3y}{dx^3}\frac{dx}{dt}$$

$$= x\frac{dy}{dx} + 3x^2\frac{d^2y}{dx^2} + x^3\frac{d^3y}{dx^3}$$

$$= \frac{dy}{dt} + 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + x^3\frac{d^3y}{dx^3}.$$

Hence the equation turn into

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = 2te^{2t}$$

$$\Rightarrow \frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 2te^{2t}$$

characteristic equation to homogeneous one:

$$r^3 - 2r^2 - r + 2 = 0 \Rightarrow r_1 = -1,\ r_2 = 1,\ r_3 = 2.$$

general solution to homogeneous one:

$$y_h(t) = c_1e^{-t} + c_2e^{t} + c_3e^{2t}.$$

Suppose particular solution is $Y_p(t) = (At^2 + Bt + C)e^{2t}$

ce^{2t} belongs to $y_h(t)$, $\Rightarrow Y_p(t) = te^{2t}(At + B)$.

Substitute $\Rightarrow$ $A = \frac{1}{3}$, $B = -\frac{8}{9}$

So general solution: $y(t) = te^{2t}\left(\frac{1}{3}t - \frac{8}{9}\right) + c_1e^{-t} + c_2e^{t} + c_3e^{2t}$

$$y(x) = x^2\ln x\left(\frac{1}{3}\ln x - \frac{8}{9}\right) + \frac{c_1}{x} + c_2x + c_3x^2. \quad \#$$

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6. (8 pts)

When $y_0 \neq 0$.

$$y' = -3y^4$$

$$\frac{dy}{y^4} = -3\,dx$$

$$-\frac{1}{3}y^{-3} = -3x + C \qquad 2'$$

$$\Rightarrow \quad y^{-3} = 9x + C$$

$$y = (9x+C)^{-\frac{1}{3}} \qquad 1'$$

$y(0) = y_0$ implies $C = y_0^{-3}$ $\quad 2'$

$$y = (9x + y_0^{-3})^{-\frac{1}{3}}.$$

Solution exists when $9x + y_0^{-3} \neq 0$.

So when $y_0 > 0$, $x \in (-\frac{1}{9}y_0^{-3}, +\infty)$

When $y_0 < 0$, $x \in (-\infty, -\frac{1}{9}y_0^{-3})$ $\quad 2'$

When $y_0 = 0$, $y = 0$ is the only solution, exists on $\mathbb{R}$.

$1'$

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7. (a) (6 pts) $y' + \frac{1+x}{x} y = \frac{1}{x}$.

$\mu = \exp\left(\int \frac{x+1}{x} dx\right) = xe^x$

Multiply μ on both sides,

$(xe^x y)' = e^x$

$\Rightarrow \quad y = \frac{1}{x} + \frac{c}{xe^x}$.

~~Since $y(1) = 1 + c/e = 0 \Rightarrow c = -e$~~

So $y = \frac{e^x - e}{xe^x}$, $\quad x \in (0, +\infty)$.

(b) (6 pts) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2x \frac{dy}{du}$

$\Rightarrow \quad \frac{2dy}{du} = \frac{1}{uy + y^3}$

$\frac{du}{dy} = 2(uy + y^3)$

$\frac{du}{dy} - 2yu = 2y^3$

$\mu = \exp\left(-\int 2y\, dy\right) = e^{-y^2}$

$\frac{d}{dy}\left(e^{-y^2} u\right) = e^{-y^2} 2y^3$

$e^{-y^2} u = -e^{-y^2}(y^2+1) + c$

$x^2 = u = -(y^2+1) + ce^{+y^2}$

$x^2 + y^2 + 1 = ce^{+y^2}$, $\quad c$ arbitrary positive constant.

(c). $f(t,y) = RHS$.

(4 pts) $f(t,y)$ Lipschitz continuous at $(0,0)$.

By Cauchy-Lipschitz theorem, $y \equiv 0$ is the only solution

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8. (a) $y'' - \frac{4}{t} y' + \frac{6}{t^2} y = 0, \quad t > 0$

(5 pts) $p(t) = -\frac{4}{t}, \quad q(t) = \frac{6}{t^2}$.

By Abel's formula,

$$W = c\exp\left(-\int p(t)\,dt\right)$$
$$= c\exp\left(\int \frac{4}{t}\,dt\right) = ct^4.$$

$$W = y_1 y_2' - y_1' y_2$$

$$\Rightarrow t^2 y_2' - 2t y_2 = ct^4.$$

$$y_2' - \frac{2}{t} y_2 = ct^2$$

$$\mu(t) = e^{-\int \frac{2}{t} dt} = t^{-2}$$

$$(t^{-2} y_2)' = c$$

$$y_2 = c_1 t^3 + c_2 t^2, \quad c_2 t^2 \text{ belongs to } y_1$$
$$\Rightarrow y_2 = t^3$$

So general solution is

$$y(t) = c_1 t^3 + c_2 t^2.$$

(b) $y_2(t) = v(t)y_1(t)$

5pt+1)

$y_2' = v'y_1 + vy_1'$

$y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$t^2 y_2'' - 4t y_2' + 6y_2$

$= t^2(v''y_1 + 2v'y_1' + vy_1'') - 4t(v'y_1 + vy_1') + 6v(t)y_1$

$= v(t^2 y_1'' - 4ty_1' + 6y_1) + t^2(v''y_1 + 2v'y_1') - 4tv'y_1$

$= 0$

since $t^2 y_1'' - 4ty_1' + 6y_1 = 0$,

$t^2(v''y_1 + 2v'y_1') - 4tv'y_1 = 0$

Substitute y_1, $\Rightarrow$ $v'' = 0$ $\Rightarrow$ $v = C_1 t + C_2$

So $y_2 = C_1 t^3 + C_2 t^2$.

$C_2 t^2$ belongs to $y_1(t) = t^2$.

So $y_2(t) = t^3$.

The general solution is $y(t) = C_1 t^2 + C_2 t^3$,

C_1, C_2 arbitrary constants. #.