MATH1050BC/1058 Assignment 6 (Answers and selected solution)

1. **Answer.**

(a) (I) Let D, R be sets
(II) Let S be a subset of D
(III) y
(IV) There exists some
(V) such that $y = h(x)$
(VI) Let U be a subset of R
(VII) D
(VIII) There exists some
(IX) U
(X) $y = h(x)$

(b) i. $\alpha = -1$.
ii. $\beta = 1$, $\gamma = \frac{5}{4}$.
iii. $\delta = -1$, $\varepsilon = \frac{5}{3}$.
iv. $\zeta = -\sqrt{2} + 1$, $\eta = \sqrt{2} + 1$.
v. $\theta = -\sqrt{2}+1$, $\kappa = -\frac{\sqrt{5}}{2}+1$, $\lambda = \frac{\sqrt{5}}{2}+1$, $\mu = \sqrt{2}+1$.
vi. $\nu = -\frac{1}{\sqrt{2}} + 1$, $\xi = \frac{1}{\sqrt{2}} + 1$.
vii. $\rho = -\frac{1}{\sqrt{2}} + 1$, $\sigma = \frac{1}{\sqrt{2}} + 1$.
viii. $\tau = -\sqrt{2}+1$, $\varphi = -\frac{1}{\sqrt{2}}+1$, $\psi = \frac{1}{\sqrt{2}}+1$, $\omega = \sqrt{2}+1$.

(c) i. $\alpha = 0, \beta = 3, \gamma = 1, \delta = -1$.
ii. $\varepsilon = -2$, $\zeta = -0.25$, $\eta = -1$, $\theta = 0$, $\kappa = 1$.

(d) i. $a = 2, b = 3$.
ii. $1, 4$.
iii. $\alpha = 0$, $\beta = 1$, $\gamma = 2$, $\delta = 4$.

2. **Answer.**

(a) i. (I) if $y \in f(S)$ then $y \in I$.
(II) Suppose $y \in f(S)$
(III) there exists some $x \in S$ such that $y = f(x)$
(IV) $y = f(x) = 2x^4 - 4 \geq 2 \cdot 1^4 - 4 = -2$
(V) Since $x \leq 2$

ii. (I) if $y \in I$ then $y \in f(S)$.
(II) Take $x = \sqrt[4]{\frac{y+4}{2}}$
(III) $x = \sqrt[4]{\frac{y+4}{2}} \geq 1$
(IV) Since $y \leq 28$, we have $\frac{y+4}{2} \leq 16$
(V) $x = \sqrt[4]{\frac{y+4}{2}} \leq 2$
(VI)

$$
\begin{aligned}
f(x) &= 2x^4 - 4 \\
&= 2\left(\sqrt[4]{\frac{y+4}{2}}\right)^4 - 4 \\
&= 2 \cdot \frac{y+4}{2} - 4 \\
&= y + 4 - 4 = y
\end{aligned}
$$

(VII) $y \in f(S)$

iii. (I) if $x \in f^{-1}(U)$ then $x \in J$.
(II) Suppose $x \in f^{-1}(U)$
(III) there exists some $y \in U$ such that $y = f(x)$
(IV) $y \in U$
(V) $2x^4 - 4 = f(x) = y \leq 4$

iv. (I) if $x \in J$ then $x \in f^{-1}(U)$
(II) Suppose $x \in J$
(III) Define $y = f(x)$
(IV) $y = f(x) = 2x^4 - 4 \leq 4$
(V) $y = f(x) = 2x^4 - 4 \geq -6$
(VI) and
(VII) $x \in f^{-1}(U)$

3. **Answer.**

(a) (I) if $z \in (g \circ f)(S)$ then $z \in g(f(S))$
(II) Suppose $z \in (g \circ f)(S)$
(III) there exists some $x \in S$ such that $z = (g \circ f)(x)$
(IV) $z = (g \circ f)(x) = g(f(x)) = g(y)$
(V) $x \in S$ and $y = f(x)$
(VI) $y \in f(S)$
(VII) $y \in f(S)$ and $z = g(y)$
(VIII) $z \in g(f(S))$
(IX) if $z \in g(f(S))$ then $z \in (g \circ f)(S)$

(X) Suppose $z \in g(f(S))$
(XI) there exists some $y \in f(S)$ such that $z = g(y)$
(XII) there exists some $x \in S$ such that $y = f(x)$
(XIII) $z = g(y) = g(f(x)) = (g \circ f)(x)$
(XIV) $x \in S$ and $z = (g \circ f)(x)$
(XV) $z \in (g \circ f)(S)$

(b) (I) Pick any subset U of B
(II) For any y, if $y \in\in f(S \cap f^{-1}(U))$ then $y \in f(S) \cap U$.
(III) Pick any object y
(IV) $y \in f(S \cap f^{-1}(U))$
(V) $x \in S \cap f^{-1}(U)$
(VI) $y = f(x)$
(VII) $x \in S$ and $x \in f^{-1}(U)$
(VIII) $y = f(x)$
(IX) $y \in f(S)$
(X) there exists some $z \in U$ such that $z = f(x)$
(XI) $f(x)$
(XII) U
(XIII) $y \in f(S)$ and
(XIV) $y \in f(S) \cap U$
(XV) For any y, if $y \in f(S) \cap U$ then $y \in\in f(S \cap f^{-1}(U))$.
(XVI) Suppose $y \in f(S) \cap U$
(XVII) $y \in f(S)$ and
(XVIII) $y \in f(S)$
(XIX) there exists some $x \in S$ such that $y = f(x)$
(XX) $y = f(x)$
(XXI) $x \in f^{-1}(U)$
(XXII) $x \in S \cap f^{-1}(U)$
(XXIII) and $y = f(x)$
(XXIV) $y \in f(S \cap f^{-1}(U))$
(XXV) $f(S \cap f^{-1}(U)) = f(S) \cap U$

(c) (I) $A = \{0, 1\}$
(II) the function $f : A \longrightarrow B$
(III) $f(0) = 2$
(IV) 0
(V) $\{2\}$
(VI) $\{2\}$
(VII) $\{0, 1\}$
(VIII) 1
(IX) $1 \notin f^{-1}(U) \cap S$
(X) $f^{-1}(U \cap f(S)) \not\subset f^{-1}(U) \cap S$

4. **Solution.**

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \dfrac{2z|z|}{1 + |z|^2}$ for any $z \in \mathbb{C}$. Let $D = \{w \in \mathbb{C} : |w| < 2\}$.

(a) Pick any $w \in f(\mathbb{C})$. By definition, there exists some $z \in \mathbb{C}$ such that $w = f(z)$.

We have $|w| = |f(z)| = \left|\dfrac{2z|z|}{1 + |z|^2}\right| = \dfrac{2|z| \cdot |z|}{1 + |z|^2} = \dfrac{2|z|^2}{1 + |z|^2} < \dfrac{2(1 + |z|^2)}{1 + |z|^2} = 2.$

(b) Pick any $w \in D$. Note that $w = 0$ or $w \neq 0$.

- (Case 1.) Suppose $w = 0$. Note that $f(0) = 0 = w$ and $0 \in \mathbb{C}$. Then $w \in f(\mathbb{C})$.
- (Case 2.) Suppose $w \neq 0$. Then $|w| > 0$.

  We have $0 < |w| < 2$. Then $\dfrac{1}{|w|(2 - |w|)}$ is well-defined as a real number. Moreover, $\dfrac{1}{|w|(2 - |w|)} > 0$. Then $\dfrac{1}{\sqrt{|w|(2 - |w|)}}$ is well-defined as a positive real number.

  Define $z = \dfrac{w}{\sqrt{|w|(2 - |w|)}}$. By definition, $z \in \mathbb{C}$. We have

$$f(z) = \frac{2\left[w/\sqrt{|w|(2 - |w|)}\right] \cdot \left|w/\sqrt{|w|(2 - |w|)}\right|}{1 + \left|w/\sqrt{|w|(2 - |w|)}\right|^2} = \frac{2w|w|/[|w|(2 - |w|)]}{1 + |w|^2/[|w|(2 - |w|)]} = \frac{2w|w|}{|w|(2 - |w|) + |w|^2} = w$$

  Then $w \in f(\mathbb{C})$.

Hence, in any case, $w \in f(\mathbb{C})$.

It follows that $D \subset f(\mathbb{C})$.

5. **Solution.**

Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \dfrac{i\bar{z}}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.

Let $H = \{z \in \mathbb{C} : \mathsf{Re}(z) > 0\}$, and $S = \{w \in \mathbb{C} : |w| = 1\}$.

(a) Pick any $w \in f(H)$.

By the definition of image sets, there exists some $z \in H$ such that $w = f(z)$.

For the same w, z, we have $|w| = |f(z)| = \left|\dfrac{i\bar{z}}{z}\right| = \dfrac{1 \cdot |\bar{z}|}{|z|} = 1$. Then $w \in S$.

We verify that $w \neq -i$:

- Suppose it were true that $w = -i$. Then for the same w, z, we have $-i = \dfrac{i\bar{z}}{z}$. Therefore $-z = \bar{z}$.

  Then $\mathsf{Re}(z) = \dfrac{z + \bar{z}}{2} = 0$.

  Since $z \in H$, we have $\mathsf{Re}(z) > 0$. Now $0 = \mathsf{Re}(z) > 0$. Contradiction arises.

Hence $w \notin \{-i\}$.

Recall that $w \in S$. Then $w \in S \backslash \{-i\}$.

It follows that $f(H) \subset S \backslash \{-i\}$.

(b) Pick any $w \in S \backslash \{-i\}$.

By definition, $w \in S$ and $w \notin \{-i\}$. Since $w \in S$, we have $|w| = 1$. Since $w \notin \{-i\}$, we have $w \neq -i$.

Define $w' = -iw$. Note that $|w'| = |w| = 1$ and $w' \neq -1$.

For the same w', there exists some $\theta \in (-\pi, \pi)$ such that $w' = \cos(\theta) + i \sin(\theta)$.

Define $z = \cos\left(-\dfrac{\theta}{2}\right) + i \sin\left(-\dfrac{\theta}{2}\right)$.

Since $\theta \in (-\pi, \pi)$, we have $-\dfrac{\theta}{2} \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Then $\cos\left(-\dfrac{\theta}{2}\right) > 0$. Hence $\mathsf{Re}(z) > 0$. Then, by definition, $z \in H$.

We have $f(z) = \dfrac{i\bar{z}}{z} = i \cdot \dfrac{\cos(-\theta/2) - i\sin(-\theta/2)}{\cos(-\theta/2) + i\sin(-\theta/2)} = i(\cos(\theta) + i\sin(\theta)) = iw' = i(-iw) = w$.

It follows that $S \backslash \{-i\} \subset f(H)$.

6. **Solution.**

(a) The statement ($\sharp$) is true. We give a proof:

Let A, B be sets, and $f : A \longrightarrow B$ be a function.
Let U, V be subsets of B. Suppose $U \subset V$.
Pick any object x. Suppose $x \in f^{-1}(U)$.
By the definition of pre-image sets, there exists some $y \in U$ such that $y = f(x)$.
Since $y \in U$ and $U \subset V$, we have $y \in V$.
Since $y \in V$ and $y = f(x)$, we have $x \in f^{-1}(V)$ according to the definition of pre-image sets.
It follows that $f^{-1}(U) \subset f^{-1}(V)$.

Acceptable answer.

Let A, B be sets, and $f : A \longrightarrow B$ be a function.
Let U, V be subsets of B. Suppose $U \subset V$.
Pick any object x. Suppose $x \in f^{-1}(U)$.
By the definition of pre-image sets, $f(x) \in U$.
Since $f(x) \in U$ and $U \subset V$, we have $f(x) \in V$.
Since $f(x) \in V$, we have $x \in f^{-1}(V)$ according to the definition of pre-image sets.
It follows that $f^{-1}(U) \subset f^{-1}(V)$.

(b) The statement ($\flat$) is false. We give a dis-proof by counter-example.

Take $A = \{0, 1\}$, $B = \{0, 1\}$. Here $0, 1$ are distinct objects.
Define the function $f : A \longrightarrow B$ by $f(0) = f(1) = 0$.
Take $U = \{0, 1\}$, $V = \{0\}$.
We have $f^{-1}(U) = f^{-1}(V) = \{0, 1\}$. Then $f^{-1}(U) \subset f^{-1}(V)$.
Note that $1 \in U$ and $1 \notin V$. Then $U \not\subset V$.

7. **Solution.**

(a) The statement ($\sharp$) is true. We give a proof:

Let A, B be sets, and $f : A \longrightarrow B$ be a function.
Let S be a subset of A.
Pick any object x. Suppose $x \in S$. Define $y = f(x)$.

Since $x \in S$ and $y = f(x)$, we have $y \in f(S)$ according to the definition of image sets.
Since $y \in f(S)$ and $y = f(x)$, we have $x \in f^{-1}(f(S))$ according to the definition of pre-image sets.
It follows that $S \subset f^{-1}(f(S))$.

(b) The statement ($\flat$) is false. We give a dis-proof by counter-example:

Take $A = \{0, 1\}, B = \{0\}$. Here $0, 1$ are distinct objects.
Define the function $f : A \longrightarrow B$ by $f(0) = f(1) = 0$.
Take $S = \{0\}$. Note that $S \subset A$.
We have $f(S) = \{0\}$ and $f^{-1}(f(S)) = \{0, 1\}$.
Note that $1 \in f^{-1}(f(S))$ and $1 \notin S$. Then $f^{-1}(f(S)) \not\subset S$.

8. **Answer.**

(a) (I) G
(II) G is a subset of A^2
(III) For any $x \in A$, $(x, x) \in G$.
(IV) For any $x, y \in A$, if $(x, y) \in G$ then $(y, z) \in G$.
(V) For any $x, y, z \in A$, if $(x, y) \in G$ and $(y, z) \in G$ then $(x, z) \in G$.
(VI) R is reflexive, symmetric and transitive.

(b) i. $(s, t) = (0, 0)$ and $(u, v) = (1, 0)$.
ii. $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = 1$.
iii. A. $(s, t) = (0, 0)$.
B. $(u, v) = (1, 2)$.
C. (I) There exist some
(II) $\in A$
(III) $(x, y) \in G$ and $(y, z) \in G$ and
(IV) 0
(V) 1
(VI) 2
(VII) $\in G$
(VIII) $\notin G$
Alternative answer. (IV) 0. (V) 1. (VI) 0. (VII) $\in G$. (VIII) $\notin G$.

(c) i. Yes. ii. No. iii. Yes. iv. Yes. v. Yes. vi. Yes. vii. No. viii. Yes. ix. No.

9. **Answer.**

(a) (I) For any $\zeta \in \mathbb{C}^*$
(II) Pick any
(III) $\in \mathbb{C}^*$
(IV) $\dfrac{\zeta}{\zeta}$
(V) $\mathbb{R}^*$
(VI) $\in E$

(b) (I) if $(\zeta, \eta) \in E$ and $(\eta, \xi) \in E$ then $(\zeta, \xi) \in E$
(II) Suppose $(\zeta, \eta) \in E$ and $(\eta, \xi) \in E$.
(III) $\dfrac{\eta}{\zeta} \in \mathbb{R}^*$
(IV) $(\eta, \xi) \in E$
(V) $\dfrac{\xi}{\zeta} \in \mathbb{R}$
(VI) $\dfrac{\eta}{\zeta} \neq 0$ and $\dfrac{\xi}{\eta} \neq 0$
(VII) $\dfrac{\xi}{\zeta} \in \mathbb{R}^*$
(VIII) (ζ, ξ)

(c) (I) symmetric
(II) For any $\zeta, \eta \in \mathbb{C}^*$, if $(\zeta, \eta) \in E$ then $(\eta, \zeta) \in E$
(III) Pick any $\zeta, \eta \in \mathbb{C}^*$
(IV) $(\zeta, \eta) \in E$
(V) $\dfrac{\eta}{\zeta} \neq 0$
(VI) $\dfrac{\zeta}{\eta} \in \mathbb{R}$
(VII) $(\eta, \zeta) \in E$
(VIII) reflexive, symmetric and transitive

10. **Answer.**

(I) a subset of $\mathbb{R}^2$
(II) for any $x \in \mathbb{R}$, $(x, x) \in E$
(III) Pick any $x \in \mathbb{R}$
(IV) $x - x = 0$
(V) $(x, x) \in E$
(VI) for any $x, y \in \mathbb{R}$, if $(x, y) \in E$ then $(y, x) \in E$
(VII) Suppose $(x, y) \in E$
(VIII) there exists some $a \in \mathbb{Q}$
(IX) $y - x = -(x - y) = -a$
(X) $a \in \mathbb{Q}$
(XI) $-a \in \mathbb{Q}$
(XII) $(y, x) \in E$
(XIII) for any $x, y, z \in \mathbb{R}$, if $(x, y) \in E$ and (y, z) then $(x, z) \in E$
(XIV) $x, y, z \in \mathbb{R}$
(XV) $(x, y) \in E$ and $(y, z) \in E$
(XVI) $x - y = a$

(XVII) Since $(y, z) \in E$, there exists some $b \in \mathbb{Q}$ such that $y - z = b$

(XVIII) Since $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$, we have $a + b \in \mathbb{Q}$

(IX) $(x, z) \in E$

11. **Answer.**

(a) (I) there exists some $x_0 \in \mathbb{R}$

(II) $(x_0, x_0) \notin G$

(III) Take $x_0 = 0$

(IV) $x_0 - x_0$

(V) $x_0 \cdot x_0$

(VI) $(x_0, x_0) \notin G$

(b) (I) there exists some $x_0, y_0 \in \mathbb{R}$

(II) $(x_0, y_0) \in G$ and $(y_0, x_0) \notin G$

(III) $x_0 = 1,\ y_0 = 0$

(IV) $1 > 0$

(V) $(x_0, y_0) \in G$

(VI) $y_0 - x_0 > x_0 y_0$

(VII) $(y_0, x_0) \notin G$

(VIII) $(x_0, y_0) \in G$ and $(y_0, x_0) \notin G$

(c) (I) there exists some $x_0, y_0, z_0 \in \mathbb{R}$

(II) $(x_0, y_0) \in G$ and $(y_0, z_0) \in G$ and $(x_0, z_0) \notin G$

(III) $z_0 = -2$

(IV) $x_0 - y_0 = -5 > -6 = x_0 y_0$

(V) by the definition of G

(VI) $(y_0, z_0) \in G$

(VII) $x_0 - z_0 = 0 \leq 4 = x_0 z_0$

(VIII) $(x_0, z_0) \notin G$

(IX) $(x_0, y_0) \in G$ and $(y_0, z_0) \in G$ and $(x_0, z_0) \notin G$

12. **Solution.**

Define the relation $S = (\mathbb{C}, \mathbb{C}, F)$ in $\mathbb{C}$ by $F = \Big\{(\zeta, \xi) \in \mathbb{C}^2 : \zeta^2 - \xi^2 = ai \text{ for some } a \in \mathbb{R}\Big\}$.

(a) Let $\zeta \in \mathbb{C}$. We have $\zeta^2 - \zeta^2 = 0 = 0 \cdot i$ and $0 \in \mathbb{R}$. Therefore $(\zeta, \zeta) \in F$.
It follows that S is reflexive.

(b) Let $\zeta, \eta \in \mathbb{C}$. Suppose $(\zeta, \eta) \in F$. Then $\zeta^2 - \eta^2 = ai$ for some $a \in \mathbb{R}$. Therefore, for the same $a \in \mathbb{R}$, we have $\eta^2 - \zeta^2 = (-a)i$ and $-a \in \mathbb{R}$. Hence $(\eta, \zeta) \in F$.
It follows that S is symmetric.

(c) Let $\zeta, \eta, \mu \in \mathbb{C}$. Suppose $(\zeta, \eta) \in F$ and $(\eta, \mu) \in F$. Since $(\zeta, \eta) \in F$, we have $\zeta^2 - \eta^2 = ai$ for some $a \in \mathbb{R}$. Since $(\eta, \mu) \in F$, we have $\eta^2 - \mu^2 = bi$ for some $b \in \mathbb{R}$. Therefore, for the same $a, b \in \mathbb{R}$, we have $\zeta^2 - \mu^2 = (a + b)i$ and $a + b \in \mathbb{R}$. Hence $(\zeta, \mu) \in F$.
It follows that S is transitive. Since S is reflexive, symmetric and transitive, S is an equivalence relation.

13. **Solution.**

Let $R = (\mathbb{R}, \mathbb{R}, G)$ be the relation in $\mathbb{R}$ defined by $G = \Big\{(x, y) \;\Big|\; x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } |y - x| \leq 1\Big\}$.

(a) Pick any $x \in \mathbb{R}$. We have $|x - x| = 0 \leq 1$. Then $(x, x) \in G$.
Hence R is reflexive.

(b) Pick any $x, y \in \mathbb{R}$. Suppose $(x, y) \in G$. Then $|x - y| \leq 1$. Therefore $|y - x| = |x - y| \leq 1$. So $(y, x) \in G$.
Hence R is symmetric.

(c) Take $x_0 = 0,\ y_0 = 1,\ z_0 = 2$. Note that $x_0, y_0, z_0 \in \mathbb{R}$.
We have $|x_0 - y_0| = 1$. Then $(x_0, y_0) \in G$.
We also have $|y_0 - z_0| = 1$. Then $(y_0, z_0) \in G$.
But $|x_0 - z_0| = 2$. Therefore $(x_0, z_0) \notin G$.
Hence R is not transitive.

(d) Since R is not transitive, R is not an equivalence relation in $\mathbb{R}$.

14. **Solution.**

Let $A = \left\{\varphi \;\middle|\; \begin{array}{l} \varphi : \mathbb{N} \longrightarrow \mathbb{R} \text{ is a function} \\ \text{and } \varphi(0) = 0. \end{array}\right\}$.

Define the relation $R = (A, A, H)$ in A by $H = \left\{(\varphi, \psi) \in A^2 : \begin{array}{l} \text{For any } n \in \mathbb{N}, \\ \varphi(n+1) - \varphi(n) \leq \psi(n+1) - \psi(n) \end{array}\right\}$.

(a) Pick any $\varphi \in A$. Pick any $n \in \mathbb{N}$. We have $\varphi(n+1) - \varphi(n) = \varphi(n+1) - \varphi(n)$. Then $(\varphi, \varphi) \in H$. It follows that R is reflexive.

(b) Pick any $\varphi, \psi, \sigma \in A$. Suppose $(\varphi, \psi) \in H$ and $(\psi, \sigma) \in H$. Pick any $x \in \mathbb{N}$. Since $(\varphi, \psi) \in H$, we have $\varphi(n+1) - \varphi(n) \leq \psi(n+1) - \varphi(n)$. Since $(\psi, \sigma) \in H$, we have $\psi(n+1) - \psi(n) \leq \sigma(n+1) - \sigma(n)$. Then $\varphi(n+1) - \varphi(n) \leq \psi(n+1) - \psi(n) \leq \sigma(n+1) - \sigma(n)$. Hence $(\varphi, \sigma) \in H$. It follows that R is transitive.

(c) Define $\varphi, \psi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\varphi(n) = 0$, $\psi(n) = n$ for any $n \in \mathbb{N}$

For any $n \in \mathbb{N}$, we have $\varphi(n+1) - \varphi(n) = 0 \leq 1 = \psi(n+1) - \psi(n)$. Then $(\varphi, \psi) \in H$.

However, $\psi(1) - \psi(0) = 1 > 0 = \varphi(1) - \varphi(0)$. It is not true that $\varphi(1) - \varphi(0) \leq \psi(1) - \psi(0)$. Then $(\psi, \varphi) \notin H$.

Hence R is not an equivalence relation in A.

15. **Solution.**

Define the relation $R = (\mathbb{Z}, \mathbb{Z}, G)$ in $\mathbb{Z}$ by $G = \left\{ (x, y) \in \mathbb{Z}^2 : \begin{array}{l} \text{There exist some } m, n \in \mathbb{N} \\ \text{such that } y = mx + n. \end{array} \right\}$.

(a) Pick any $x \in \mathbb{Z}$. We have $x = 1 \cdot x + 0$ and $1, 0 \in \mathbb{N}$. Then $(x, x) \in G$. It follows that R is reflexive.

(b) Pick any $x, y, z \in \mathbb{Z}$. Suppose $(x, y) \in G$ and $(y, z) \in G$.

Since $(x, y) \in G$, there exist some $m, n \in \mathbb{N}$ such that $y = mx + n$. Since $(y, z) \in G$, there exist some $p, q \in \mathbb{N}$ such that $z = py + q$.

Now, for the same $m, n, p, q \in \mathbb{N}$, we have $z = py + q = p(mx + n) + q = (pm)x + (pn + q)$. Also, $pm \in \mathbb{N}$ and $pn + q \in \mathbb{N}$. Then $(x, z) \in G$.

It follows that R is transitive.

(c) We verify that R is not symmetric:

- Take $x_0 = -1$, $y_0 = 1$. Note that $x_0, y_0 \in \mathbb{Z}$. We have $y_0 = 1 = 0 \cdot (-1) + 1 = 0 \cdot x_0 + 1$, and $0, 1 \in \mathbb{N}$. Then $(x_0, y_0) \in G$.

  Suppose it were true that $(y_0, x_0) \in G$. Then there would exist some $p, q \in \mathbb{N}$ such that $x_0 = py_0 + q$.

  Then $-1 = p \cdot 1 + q = p + q$. Since $p, q \in \mathbb{N}$, we have $p + q \geq 0$. Then $-1 = p + q \geq 0$. Contradiction arises.

  It follows that $(y_0, x_0) \notin G$.

Hence R is not symmetric. It follows that R is not an equivalence relation in $\mathbb{Z}$.

16. **Solution.**

Let A be the set of all real-valued functions with domain $[1, +\infty)$.

Let R be the relation in A with graph E given by

$$E = \left\{ (f, g) \;\middle|\; \begin{array}{l} f \in A \text{ and } g \in A \\ \text{and (there exist some positive real numbers } \alpha, K \\ \text{such that } \displaystyle\lim_{t \longrightarrow +\infty} \frac{f(t) - Kg(t)}{t^\alpha} \text{ exists and equals } 0). \end{array} \right\}.$$

(a) Pick any $f \in A$.

Take $\alpha = 1$, $K = 1$. Note that α, K are positive real numbers.

For any $t \in [1, +\infty)$, we have $\dfrac{f(t) - Kf(t)}{t^\alpha} = \dfrac{f(t) - 1 \cdot f(t)}{t^1} = 0$.

Then $\displaystyle\lim_{t \longrightarrow +\infty} \frac{f(t) - Kf(t)}{t^\alpha}$ exists and equals 0.

Therefore $(f, f) \in E$.

(b) Let $f, g \in A$. Suppose $(f, g) \in E$.

Then there exist some positive real numbers α, K such that $\displaystyle\lim_{t \longrightarrow +\infty} \frac{f(t) - Kg(t)}{t^\alpha}$ exists and equals 0.

Take $\beta = \alpha$, $L = \dfrac{1}{K}$. (Since K is non-zero, L is well-defined as a real number.) Note that β, L are positive real numbers.

For any $t \in [1, +\infty)$, we have $\dfrac{g(t) - Lf(t)}{t^\beta} = -\dfrac{1}{K} \cdot \dfrac{f(t) - Kg(t)}{t^\alpha}$.

Then $\displaystyle\lim_{t \longrightarrow +\infty} \frac{g(t) - Lf(t)}{t^\beta}$ exists and

$$\lim_{t \longrightarrow +\infty} \frac{g(t) - Lf(t)}{t^\beta} = \lim_{t \longrightarrow +\infty} -\frac{1}{K} \cdot \frac{f(t) - Kg(t)}{t^\alpha} = 0.$$

(c) We verify that R is transitive.

- Let $f, g, h \in A$. Suppose $(f, g) \in E$ and $(g, h) \in E$.

  Since $(f, g) \in E$, there exist some positive real numbers α, K such that $\displaystyle\lim_{t \longrightarrow +\infty} \frac{f(t) - Kg(t)}{t^\alpha}$ exists and equals 0.

Since $(g,h) \in E$, there exist some positive real numbers β, L such that $\lim\limits_{t \longrightarrow +\infty} \dfrac{g(t) - Lh(t)}{t^\beta}$ exists and equals 0.
Take $\gamma = \alpha + \beta$, $M = KL$. Note that γ, M are positive real numbers.
For any $t \in [1, +\infty)$, we have

$$\frac{f(t) - Mh(t)}{t^\gamma} = \frac{f(t) - KLh(t)}{t^\alpha t^\beta} = \frac{1}{t^\beta} \cdot \frac{f(t) - Kg(t)}{t^\alpha} + \frac{K}{t^\alpha} \cdot \frac{g(t) - Lh(t)}{t^\beta}$$

Then $\lim\limits_{t \longrightarrow +\infty} \dfrac{f(t) - Mh(t)}{t^\gamma}$ exists and

$$\lim_{t \longrightarrow +\infty} \frac{f(t) - Mh(t)}{t^\gamma} = \lim_{t \longrightarrow +\infty} \frac{1}{t^\beta} \cdot \frac{f(t) - Kg(t)}{t^\alpha} + \lim_{t \longrightarrow +\infty} \frac{K}{t^\alpha} \cdot \frac{g(t) - Lh(t)}{t^\beta} = 0 + 0 = 0.$$

Since R is reflexive, symmetric and transitive, R is an equivalence relation in A.

17. **Answer.**

(a) (I) there exists some
(II) f is a bijective function
(III) there exists some
(IV) f is an injective function
(V) there is some injective function from A to B
(VI) there is no bijective function from A to B

(b) i. Suppose A is a set. Then we say A is infinite if $\mathbb{N} \lesssim A$.
ii. Suppose A is a set. Then we say A is countable if $A \lesssim \mathbb{N}$.

(c) i. Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.
ii. Suppose A is a set. Then $A < \mathfrak{P}(A)$.

(d) These sets are of cardinality equal to $\mathbb{N}$:

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{N}^3$, $\mathsf{Map}(\{0,1\}, \mathbb{N})$.

These sets are of cardinality equal to $\mathfrak{P}(\mathbb{N})$:

$\mathbb{R} \backslash \mathbb{Q}$, $[0,1]$, $\mathfrak{P}(\mathbb{N})$, $\mathsf{Map}(\mathbb{N}, \{0,1\})$, $\mathbb{C}$, $\mathbb{N} \times \mathbb{R}$.

These sets are of cardinality equal to $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$:

$\mathfrak{P}(\mathbb{R})$, $\mathsf{Map}(\mathbb{R}, \{0,1\})$.

(e) i. True.
ii. True.
iii. False.
iv. False.
v. True.

18. (a) **Answer.**

(I) $C \cap C'$
(II) $\emptyset$
(III) g, g' are bijective functions
(IV) $(C \cup C', D \cup D', G \cup G')$ is a bijective function

(b) i. (I) H is a subset of $J \times K$
(II) there exists some $y \in K$ such that $(x,y) \in H$
(III) $x \in J$
(IV) $y = 1 - x$
(V) $x \in J$
(VI) $y = 1 - x \leq 1 - 0 = 1$
(VII) $y = 1 - x > 1 - 1 = 0$
(VIII) $y \in K$
(IX) $(x,y) \in H$
(X) if $(x,y) \in H$ and $(x,z) \in H$ then $y = z$
(XI) Pick any $x \in J$, $y, z \in K$
(XII) $(x,y) \in H$ and $(x,z) \in H$
(XIII) Since $(x,y) \in H$
(XIV) $x + z = 1$
(XV) $z = 1 - x$
(XVI) $y = 1 - x = z$
(XVII) for any $y \in K$, there exists some $x \in J$
(XVIII) $(x,y) \in H$
(XIX) Pick any $y \in K$
(XX) $0 < y \leq 1$
(XXI) $y > 0$
(XXII) $x = 1 - y \geq 1 - 1 = 0$
(XXIII) $(x,y) \in H$
(XXIV) for any $x, w \in J$, for any $y \in K$
(XXV) then $x = w$
(XXVI) Suppose $(x,y) \in H$ and $(w,y) \in H$
(XXVII) Since $(x,y) \in H$
(XXVIII) $w + y = 1$
(XXIX) $x = 1 - y = w$
(XXX) h is a function
(XXXI) h is surjective
(XXXII) h is injective
(XXXIII) J is of cardinality equal to K

ii. **Solution.**
Let $L = (0,1)$, $M = \{0\}$, $N = \{1\}$.

Note that $J = L \cup M$, $K = L \cup N$, and $L \cap M = \emptyset$, $L \cap N = \emptyset$.
Let $D = \{(x, x) \mid x \in L\}$. The identity function id_L is a bijective function from L to L with graph D.
Note that $M \times N = \{(0, 1)\}$. The relation $(M, N, M \times N)$ is a bijective function from M to N with graph $M \times N$.
Define $F = D \cup (M \times N)$, and define $f = (J, K, F)$.
By the Glueing Lemma, f is a bijective function from J to K with graph F.
It follows that $J \equiv K$.

19. **Answer.**
Let $J = [0, 1)$, $L = (0, 1)$, $M = [0, +\infty)$, $N = (0, +\infty)$.

(a) A bijective function from J to M is $\varphi : J \longrightarrow M$, given by $\varphi(x) = -1 - \dfrac{1}{x-1}$ for any $x \in J$. It follows that $J{\sim}M$.

A bijective function from L to N is $\psi : L \longrightarrow N$, given by $\psi(x) = -1 - \dfrac{1}{x-1}$ for any $x \in L$. It follows that $L{\sim}M$.

(b) A bijective function from $\mathbb{R}$ to $(-1, 1)$ is $\alpha : \mathbb{R} \longrightarrow (-1, 1)$, given by $\alpha(x) = \dfrac{1 - e^{-x}}{1 + e^{-x}}$ for any $x \in \mathbb{R}$. It follows that $\mathbb{R}{\sim}(-1, 1)$.

A bijective function from $(-1, 1)$ to $(0, 1)$ is $\beta : (-1, 1) \longrightarrow (0, 1)$, given by $\beta(x) = \dfrac{x+1}{2}$ for any $x \in (-1, 1)$. Now $\alpha^{-1} \circ \beta^{-1}$ a bijective function from L to $\mathbb{R}$. It follows that $L{\sim}\mathbb{R}$.

20. **Solution.**
Let $D = \Big\{z \in \mathbb{C} : |z| < 1\Big\}$, $H = \Big\{w \in \mathbb{C} : \mathsf{Im}(w) > 0\Big\}$.

Define $F = \left\{(z, w) \,\middle|\, z \in D \text{ and } w \in H \text{ and } w = \dfrac{z+i}{iz+1}\right\}$, and $f = (D, H, F)$.

(a)
- By definition, $F \subset D \times H$. Then f is a relation from D to H with graph F.
- Pick any $z \in D$. Since $|z| < 1$, we have $iz + 1 = i(z - i) \neq 0$. Define $w = \mathsf{Im}\left(\dfrac{z+i}{iz+1}\right)$.

  $$\mathsf{Im}(w) = \mathsf{Im}\left(\frac{z+i}{iz+1}\right) = \frac{1}{2i}\left[\frac{z+i}{iz+1} - \overline{\left(\frac{z+i}{iz+1}\right)}\right] = \dots = \frac{1 - |z|^2}{|z-i|^2}.$$

  Since $|z| < 1$, $1 - |z|^2 > 0$. Then $\mathsf{Im}(w) = \dfrac{1 - |z|^2}{|z-i|^2} > 0$. Therefore $w \in H$.

  We have $(z, w) \in F$.
- Pick any $z \in D$. Pick any $w, w' \in H$. Suppose $(z, w) \in F$ and $(z, w') \in F$.
  By definition, $w = \dfrac{z+i}{iz+1}$ and $w' = \dfrac{z+i}{iz+1}$. Then $w = w'$.

Therefore f is a function.

(b) Note that $f(z) = \dfrac{z+i}{iz+1}$ for any $z \in D$.
We verify that f is bijective:

- Pick any $z, z' \in D$. Suppose $f(z) = f(z')$. Then $\dfrac{z+i}{iz+1} = \dfrac{z'+i}{iz'+1}$.
  Therefore $izz' - z' + z + i = (z + i)(iz' + 1) = (z' + i)(iz + 1) = izz' - z + z' + i$. Hence $z = z'$.
  It follows that f is injective.
- Pick any $w \in H$. Since $\mathsf{Im}(w) > 0$, we have $-iw + 1 = -i(w + i) \neq 0$. Define $z = \dfrac{w - i}{-iw + 1}$.

  $$|z|^2 = z\bar{z} = \left(\frac{w-i}{-iw+1}\right)\overline{\left(\frac{w-i}{-iw+1}\right)} = \dots = \frac{|w|^2 + 1 - 2\mathsf{Im}(w)}{|w|^2 + 1 + 2\mathsf{Im}(w)}.$$

  Then $1 - |z|^2 = 1 - \dfrac{|w|^2 + 1 - 2\mathsf{Im}(w)}{|w|^2 + 1 + 2\mathsf{Im}(w)} = \dfrac{4\mathsf{Im}(w)}{|w|^2 + 1 + 2\mathsf{Im}(w)}$.
  Since $\mathsf{Im}(w) > 0$, $1 - |z|^2 > 0$. Then $|z| < 1$. Therefore $z \in D$.
  We have $f(z) = \dfrac{z+i}{iz+1} = \dfrac{(w-i)/(-w+1) + i}{i(w-i)/(-iw+1) + 1} = \dfrac{(w-i) + i(-iw+1)}{i(w-i) + (-iw+1)} = \dfrac{2w}{2} = w$.
  It follows that f is surjective.

It follows that $D{\sim}H$.

Remark. This bijective function $f : D \longrightarrow H$ is a very special bijective function from the 'unit disc' to the 'upper-half plane' in the Argand plane. It is an example of 'fractional linear transformations'. But this belongs to another story; you will find out more about it in your *complex variables* course.

21. **Solution.**
Let $I = (0, +\infty)$, $J = [-1, 1]$.

(a) Pick any $a \in I$. We have $a > 0$. Then $a + 1 > 1$. Therefore $0 < \dfrac{1}{a+1} < 1$. Hence $\dfrac{1}{a+1} \in J$.

(b) Pick any $x, w \in I$. Suppose $g(x) = g(w)$. Then $\dfrac{1}{x+1} = \dfrac{1}{w+1}$. Therefore $x + 1 = w + 1$. Hence $x = w$.
It follows that g is injective.

(c) For any $y \in J$, we have $-1 \leq y \leq 1$, $2 \leq y + 3 \leq 4$ and hence $y + 3 \in I$.
Define the function $h : J \longrightarrow I$ by $h(y) = y + 3$ for any $y \in J$.
We verify that h is injective:

- Pick any $y, z \in J$. Suppose $h(y) = h(z)$. Then $y + 3 = z + 3$. Therefore $y = z$.

There is an injective function from I to J, namely g. Then $I \lesssim J$.
There is an injective function from J to I, namely h. Then $J \lesssim I$.
According to the Schröder-Bernstein Theorem, we have $I \sim J$.

22. **Solution.**
Let $A = [-1, 1]$, $B = (-4, -2] \cup [2, 4)$.

(a)
- Pick any $x \in A$. We have $-1 \leq x \leq 1$. Then $2 \leq \dfrac{x}{2} + \dfrac{5}{2} \leq 3$. Therefore $\dfrac{x}{2} + \dfrac{5}{2} \in [2, 3] \subset B$.

Define the function $f : A \longrightarrow B$ by $f(x) = \dfrac{x}{2} + \dfrac{5}{2}$ for any $x \in A$.
We verify that f is injective:

- Pick any $x, w \in A$. Suppose $f(x) = f(w)$. Then $\dfrac{x}{2} + \dfrac{5}{2} = \dfrac{w}{2} + \dfrac{5}{2}$. We have $x = w$.

(b)
- Pick any $y \in B$. We have $-4 \leq y \leq 4$. Then $-1 \leq \dfrac{y}{4} \leq 1$. Therefore $\dfrac{y}{4} \in A$.

Define the function $g : B \longrightarrow A$ by $g(y) = \dfrac{y}{4}$ for any $y \in B$.
We verify that g is injective:

- Pick any $y, z \in B$. Suppose $g(y) = g(z)$. Then $\dfrac{y}{4} = \dfrac{z}{4}$. We have $y = z$.

There is an injective function from A to B, namely, $f : A \longrightarrow B$. Then $A \lesssim B$.
There is an injective function from B to A, namely, $g : B \longrightarrow A$. Then $B \lesssim A$.
According to the Schröder-Bernstein Theorem, since $A \lesssim B$ and $B \lesssim A$, we have $A \sim B$.

23. **Solution.**
Let $A = [1010, 1050] \backslash \{1030\}$ and $B = (2040, 2050) \cup ([2060, +\infty) \cap \mathbb{Q})$.

(a) Define the function $f : A \longrightarrow B$ by $f(x) = \dfrac{x}{1050} + 2040$ for any $x \in A$.
Note that $\dfrac{x}{1050} + 2040 \in B$ for any $x \in A$. Then f is well-defined as a function.
We verify that f is injective:

* Pick any $x, w \in A$. Suppose $f(x) = f(w)$. Then $\dfrac{x}{1050} + 2040 = \dfrac{w}{1050} + 2040$. Therefore $\dfrac{x}{1050} = \dfrac{w}{1050}$. Hence $x = w$.

Therefore $A \lesssim B$.

(b)
- Define the function $g : B \longrightarrow A$ by $g(y) = \dfrac{1}{y} + 1010$ for any $y \in B$.
  Note that $\dfrac{1}{y} + 1010 \in A$ for any $y \in B$. Then g is well-defined as a function.
  We verify that g is injective:
  * Pick any $y, z \in B$. Suppose $g(y) = g(z)$. Then $\dfrac{1}{y} + 1010 = \dfrac{1}{z} + 1010$. Therefore $\dfrac{1}{y} = \dfrac{1}{z}$. Hence $y = z$.

Therefore $B \lesssim A$.

- By Schröder-Bernstein Theorem, since $A \lesssim B$ and $B \lesssim A$, we have $A \sim B$.

24. **Solution.**

Let $D = \left\{\zeta \in \mathbb{C} : |\zeta| \leq 1\right\}$. Define $F = \left\{(z, w) \,\middle|\, z \in \mathbb{C} \text{ and } w \in D \text{ and } w = \dfrac{iz|z|}{1+|z|+|z|^2}\right\}$. Note that $F \subset \mathbb{C} \times D$.

Define $f = (\mathbb{C}, D, F)$.

(a) - We verify that for any $z \in \mathbb{C}$, there exists some $w \in D$ such that $(z, w) \in F$:

Pick any $z \in \mathbb{C}$.

We have $|z|^2 < 1 + |z| + |z|^2$.

Then $\left|\dfrac{iz|z|}{1+|z|+|z|^2}\right| = \dfrac{|z|^2}{1+|z|+|z^2|} < 1$.

Define $w = \dfrac{iz|z|}{1+|z|+|z|^2}$.

By definition, $w \in D$.

Therefore $(z, w) \in F$.

- We verify that for any $z \in \mathbb{C}$, for any $w, v \in D$, if $(z, w) \in F$ and $(z, v) \in F$ then $w = v$:

Pick any $z \in \mathbb{C}$, $w, v \in D$. Suppose $(z, w) \in F$ and $(z, v) \in F$.

Then $w = \dfrac{iz|z|}{1+|z|+|z|^2}$ and $v = \dfrac{iz|z|}{1+|z|+|z|^2}$.

Therefore $w = v$.

Hence f is a function.

(b) We claim that f is injective. We verify that for any $z, u \in \mathbb{C}$, if $f(z) = f(u)$ then $z = u$:

Pick any $z, u \in \mathbb{C}$. Suppose $f(z) = f(u)$.

Then $\dfrac{iz|z|}{1+|z|+|z|^2} = \dfrac{iu|u|}{1+|u|+|u|^2}$.

Therefore $\dfrac{|z|^2}{1+|z|+|z|^2} = \left|\dfrac{iz|z|}{1+|z|+|z|^2}\right| = |f(z)| = |f(u)| = \left|\dfrac{iu|u|}{1+|u|+|u|^2}\right| = \dfrac{|u|^2}{1+|u|+|u|^2}$.

Then $|z|^2 + |z|^2|u| + |z|^2|u|^2 = |z|^2(1+|u|+|u|^2) = |u|^2(1+|z|+|z|^2) = |u|^2 + |u|^2|z| + |u|^2|z|^2$.

Therefore $(|z|-|u|)(|z|+|u|+|u||z|) = 0$. Hence $|z| = |u|$ or $|z|+|u|+|z||u| = 0$. If $|z|+|u|+|u||z| = 0$ then $|z| = |u| = 0$. Hence in any case $|z| = |u|$.

Now $\dfrac{iz|z|}{1+|z|+|z|^2} = f(z) = f(u) = \dfrac{iu|u|}{1+|u|+|u|^2} = \dfrac{iu|z|}{1+|z|+|z|^2}$.

Then $(z-u)|z| = 0$. Therefore $z = u$ or $|z| = 0$. If $|z| = 0$ then $|u| = 0$ and $z = u = 0$. Hence in any case $z = u$.

(c) We claim that f is not surjective. We verify that there exists some $w_0 \in D$ such that for any $z \in \mathbb{C}$, $f(z) \neq w_0$.

Take $w_0 = 1$. Note that $w_0 \in D$.

Suppose it were true that there existed some $z_0 \in \mathbb{C}$ such that $f(z_0) = w_0$.

Then we would have $1 = w_0 = \dfrac{iz_0|z_0|}{1+|z_0|+|z_0|^2}$.

Therefore $1 = \left|\dfrac{iz_0|z_0|}{1+|z_0|+|z_0|^2}\right| = \dfrac{|z_0|^2}{1+|z_0|+|z_0|^2} < 1$.

Contradiction arises.

Alternative argument.

Take $w_0 = 1$. Note that $w_0 \in D$.

Pick any $z \in \mathbb{C}$.

We have $f(z) = \dfrac{iz|z|}{1+|z|+|z|^2}$.

Note that $|z|^2 < 1 + |z| + |z|^2$.

Then we would have $|f(z)| = \left|\dfrac{iz|z|}{1+|z|+|z|^2}\right| = \dfrac{|z|^2}{1+|z|+|z|^2} < 1 = w_0$.

Therefore $f(z) \neq w_0$.

(d) By Part (a), (b), there is an injective function from $\mathbb{C}$ to D, namely, $f : \mathbb{C} \longrightarrow D$ given by $f(z) = \dfrac{iz|z|}{1+|z|+|z|^2}$ for any $z \in \mathbb{C}$. Then $\mathbb{C} \lesssim D$.

Note that $D \subset \mathbb{C}$. The inclusion function $\iota_D : D \longrightarrow \mathbb{C}$ given by $g(z) = z$ for any $z \in D$ is injective. Then $D \lesssim \mathbb{C}$.

By the Schröder-Bernstein Theorem, we have $D \sim \mathbb{C}$.