

Advice.

1. The questions in this assignment are mostly about image sets and preimage sets, reflexivity, symmetry, transitivity, equivalence relations, equipotence and subpotence.

Do familiarize yourself with the corresponding material available in the course homepage before trying the questions.

2. (a) Questions (1)-(7) are about image sets and pre-image sets.
(b) Questions (8)-(16) are about reflexivity, symmetry, transitivity, and equivalence relations.
(c) Questions (17)-(24) are about equipotence and subpotence.
3. Questions which require more thought and/or work and/or tricks and/or organization and/or ... than the 'unlabelled' questions are labelled by $\diamond, \clubsuit, \heartsuit, \spadesuit$ in ascending order of overall difficulty level.

Instructions.

1. Any work submitted by you must be written on **A4-size sheets** and must be appropriately binded.
Your name and student ID, as in your student card, and the **code** of the section to which you are registered must be written at the upper right corner of the first page of your submission.

2. Mandatory work, for assessment purpose.

You are **required** to submit work on Questions (1), (2), (3), (6) for course assessment purpose.

* * *

1. (a) Fill in the blanks in the passage below so as to give the definitions for the notions of *image set of a set under a function*, *pre-image set of a set under a function*:

- _____ (I) _____ and $h : D \longrightarrow R$ be a function.
- _____ (II) _____. The **image set of the set S under the function h** is defined to be the set $\left\{ \text{_____ (III) } \in R : \text{_____ (IV) } x \in S \text{ _____ (V) } \right\}$. It is denoted by $h(S)$.
 - _____ (VI) _____. The **pre-image set of the set U under the function h** is defined to be the set $\left\{ x \in \text{_____ (VII) } : \text{_____ (VIII) } y \in \text{_____ (IX) } \text{ such that } \text{_____ (X) } \right\}$. It is denoted by $h^{-1}(U)$

- (b) There is no need to give any justifications for your answers.

Let $f : \mathbb{R} \setminus \{0, 2\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{2}{x(x-2)} + 1$ for any $x \in \mathbb{R} \setminus \{0, 2\}$.

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \rho, \sigma, \tau, \varphi, \psi, \omega$, so that the set equalities below hold.

- | | |
|--|---|
| i. $f((0, 2)) = (-\infty, \alpha]$. | v. $f^{-1}([3, 9]) = [\theta, \kappa] \cup [\lambda, \mu]$. |
| ii. $f([4, +\infty)) = (\beta, \gamma]$. | vi. $f^{-1}([-3, 0]) = [\nu, \xi]$. |
| iii. $f((1, 3) \setminus \{2\}) = (-\infty, \delta) \cup (\varepsilon, +\infty)$. | vii. $f^{-1}([-3, 1]) = [\rho, \sigma]$. |
| iv. $f^{-1}(\{3\}) = \{\zeta, \eta\}$. | viii. $f^{-1}([-3, 3]) = (-\infty, \tau] \cup [\varphi, \psi] \cup [\omega, +\infty)$. |

- (c) There is no need to give any justifications for your answers.

Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} x^{-2} & \text{if } x < -1 \\ -1 & \text{if } x = -1 \\ -x & \text{if } -1 < x < 0 \\ 2 & \text{if } x = 0 \\ 2x^2 + 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 + x^{-1} & \text{if } x > 1 \end{cases}.$$

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa$, so that the set equalities below hold.

- $g(\mathbb{R}) = ([\alpha, \beta) \setminus \{\gamma\}) \cup \{\delta\}$.
- $g^{-1}([0.25, 3]) = ([\varepsilon, \zeta] \setminus \{\eta\}) \cup ([\theta, +\infty) \setminus \{\kappa\})$.

(d) *There is no need to give any justifications for your answers.*

Let $a, b \in \mathbb{R}$, and $h : [0, 5] \rightarrow \mathbb{R}$ be the function defined by

$$h(x) = \begin{cases} -\frac{12}{(x+1)(x-3)} & \text{if } 0 \leq x < 3 \\ a & \text{if } x = 3 \\ -(x-3)(x-5) & \text{if } 3 < x \leq 5 \text{ and } x \neq 4 \\ b & \text{if } x = 4 \end{cases}.$$

Suppose $h(3) < h(4)$. Further suppose that $h^{-1}(\{2\}) \neq \emptyset$ and $h^{-1}(\{3\})$ has exactly two elements.

- What are the respective values of a, b ?
- Name all two elements of $h^{-1}(\{3\})$.
- What are the numbers $\alpha, \beta, \gamma, \delta$ for which the set equality $h([2, 4)) = (\alpha, \beta) \cup \{\gamma\} \cup [\delta, +\infty)$ holds?

2. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = 2x^4 - 4$ for any $x \in \mathbb{R}$.

Denote by $(A), (B), (C), (D)$ the statements below:—

- (A) $f([1, 2]) \subset [-2, 28]$. (C) $f^{-1}([-6, 4]) \subset [-\sqrt{2}, \sqrt{2}]$.
 (B) $[-2, 28] \subset f([1, 2])$. (D) $[-\sqrt{2}, \sqrt{2}] \subset f^{-1}([-6, 4])$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs for the statements $(A), (B), (C), (D)$ in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

i. Here we prove the statement (A) :—

Write $S = [1, 2]$, $I = [-2, 28]$.
 [We want to verify the statement $(\dagger)$: ‘for any y , _____ (I) _____.’]
 Pick any y . _____ (II) _____. Then by the definition of $f(S)$, _____ (III) _____.
 For the same x , since $x \in S$, we have $1 \leq x \leq 2$.
 Since $x \geq 1$, we have _____ (IV) _____.
 _____ (V) _____ we have $y = f(x) = 2x^4 - 4 \leq 2 \cdot 2^4 - 4 = 28$.
 Therefore $-2 \leq y \leq 28$. Hence $y \in [-2, 28] = I$.

ii. Here we prove the statement (B) :—

Write $S = [1, 2]$, $I = [-2, 28]$.
 [We want to verify the statement $(\ddagger)$: ‘for any y , _____ (I) _____.’]
 Pick any y . Suppose $y \in I$. Then $-2 \leq y \leq 28$.
 [We want to verify that for this y , there exists some $x \in S$ such that $y = f(x)$.]
 _____ (II) _____. We verify that $x \in S$:
 * Since $y \geq -2$, we have $\frac{y+4}{2} \geq 1$. Then _____ (III) _____.
 _____ (IV) _____. Then _____ (V) _____.
 Therefore $1 \leq x \leq 2$. Hence $x \in [1, 2] = S$.
 For the same x , we have _____ (VI) _____.
 Then, for the same x, y , we have $x \in S$ and $y = f(x)$.
 Hence by the definition of $f(S)$, _____ (VII) _____.

iii. Here we prove the statement (C) :—

Write $U = [-6, 4]$, $J = [-\sqrt{2}, \sqrt{2}]$.
 [We want to verify the statement $(\dagger)$: ‘for any x , _____ (I) _____.’]
 Pick any x . _____ (II) _____. Then by the definition of $f^{-1}(U)$, _____ (III) _____.
 For the same y , since _____ (IV) _____, we have $-6 \leq y \leq 4$.
 Since $y \geq -6$, we have $2x^4 - 4 = f(x) = y \geq -6$. Then $x^4 \geq -1$. (This provides no information other than re-iterating ‘ $x \in \mathbb{R}$ ’.)
 Since $y \leq 4$, we have _____ (V) _____. Then $x^4 \leq 4$. Since $x \in \mathbb{R}$, we have $-\sqrt{2} \leq x \leq \sqrt{2}$. Then $x \in [-\sqrt{2}, \sqrt{2}] = J$.

iv. Here we prove the statement (D):—

Write $U = [-6, 4]$, $J = [-\sqrt{2}, \sqrt{2}]$.
 [We want to verify the statement (‡): ‘for any x , _____ (I) .’]
 Pick any x . _____ (II) . Then $-\sqrt{2} \leq x \leq \sqrt{2}$.

 [We want to verify that for this x , there exists some $y \in U$ such that $y = f(x)$.]
 _____ (III) . We verify that $y \in U$:

 * Since $-\sqrt{2} \leq x \leq \sqrt{2}$, we have $x^4 \leq 4$. Then _____ (IV) .
 Since $x \in \mathbb{R}$, we have $x^4 \geq 0 \geq -1$. Then _____ (V) .
 Therefore $-6 \leq y \leq 4$. Hence $y \in [-6, 4] = U$.
 Then, for the same x, y , we have $y = f(x)$ _____ (VI) $y \in U$.
 Hence by the definition of $f^{-1}(U)$, _____ (VII) .

3. Denote by (J), (K), (L) the respective statements below:—

- (J) Let A, B, C be sets, and $f : A \rightarrow B$, $g : B \rightarrow C$ be functions. Suppose S is a subset of A . Then $(g \circ f)(S) = g(f(S))$.
 (K) Suppose A, B are sets, and $f : A \rightarrow B$ is a function. Then for any subset S of A , for any subset U of B , $f(S \cap f^{-1}(U)) = f(S) \cap U$.
 (L) Suppose A, B are sets, and $f : A \rightarrow B$ is a function. Then for any subset S of A , for any subset U of B , $f^{-1}(U \cap f(S)) \subset f^{-1}(U) \cap S$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs for the statements (J), (K) and the *partially completed* dis-proof against the statement (L) in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof/dis-proof for/against each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

(a) Here we prove the statement (J):—

Let A, B, C be sets, and $f : A \rightarrow B$, $g : B \rightarrow C$ be functions. Suppose S is a subset of A .

- [We want to deduce $(g \circ f)(S) \subset g(f(S))$.
 This amounts to verifying the statement (‡): ‘For any z , _____ (I) .’]
 Pick any object z . _____ (II) . [Ask: Is it true that $z \in g(f(S))$?]
 For the same z , by the definition of $(g \circ f)(S)$, _____ (III)
 Define $y = f(x)$. By the definition of compositions, _____ (IV) .
 For the same x, y , since _____ (V) , we have _____ (VI) by the definition of $f(S)$.
 For the same y, z , since _____ (VII) , we have _____ (VIII) by the definition of $g(f(S))$.
 It follows that $(g \circ f)(S) \subset g(f(S))$.
- [We want to deduce $g(f(S)) \subset (g \circ f)(S)$.
 This amounts to verifying the statement (‡): ‘For any z , _____ (IX) .’]
 Pick any object z . _____ (X) . [Ask: Is it true that $z \in (g \circ f)(S)$?]
 For the same z , by the definition of $g(f(S))$, _____ (XI) .
 For the same y , by the definition of $f(S)$, _____ (XII) .
 For the same x, y, z , we have _____ (XIII) , by the definition of composition.
 Since _____ (XIV) , we have _____ (XV) by the definition of $(g \circ f)(S)$.
 It follows that $g(f(S)) \subset (g \circ f)(S)$.
 Hence $(g \circ f)(S) = g(f(S))$.

(b) Here we prove the statement (K):—

Suppose A, B are sets, and $f : A \rightarrow B$ is a function. Pick any subset S of A . (I) .

- [We want to deduce $f(S \cap f^{-1}(U)) \subset f(S) \cap U$.
This amounts to verifying the statement ($\dagger$): ‘ (II) ’]
(III) . Suppose (IV) . Then by the definition of image set, there exists some
(V) such that (VI) .

We have $x \in S \cap f^{-1}(U)$. Then by the definition of intersection, we have (VII) .—(*)

In particular, $x \in S$. We have $x \in S$ and (VIII) . Then, by the definition of image set, we have
(IX) .—(**)

By (*), we also have $x \in f^{-1}(U)$. Then, by the definition of pre-image set, (X) . Now
 $y =$ (XI) $= z \in$ (XII) .—(***)

Now by (**), (***), we have (XIII) $y \in U$.

Hence, by the definition of intersection, we have (XIV) .

It follows that $f(S \cap f^{-1}(U)) \subset f(S) \cap U$.

- [We want to deduce $f(S) \cap U \subset f(S \cap f^{-1}(U))$.
This amounts to verifying the statement ($\ddagger$): ‘ (XV) ’]
Pick any object y . (XVI) . Then, by the definition of intersection, (XVII)
 $y \in U$.
In particular, (XVIII) . Then, by the definition of image set, (XIX) . —(*)
For the same y , we have (XX) and $y \in U$. Then, by the definition of pre-image set, we have
(XXI) .—(**)
By (*), (**), we have $x \in S$ and $x \in f^{-1}(U)$. Hence, by the definition of intersection, we have
(XXII) .
Now we have $x \in S \cap f^{-1}(U)$ (XXIII) .
Therefore, by the definition of image set, we have (XXIV) .
It follows that $f(S) \cap U \subset f(S \cap f^{-1}(U))$.
Hence (XXV) .

(c) Here we dis-prove the statement (L):—

Regard 0, 1, 2, 3 as pairwise distinct objects. Take (I) , $B = \{2, 3\}$.
Define (II) by (III) , $f(1) = 2$.
Take $S = \{0\}$, $U = B$.
Note that $f^{-1}(U) = \{0, 1\}$. Then $f^{-1}(U) \cap S = \{$ (IV) $\}$.
Note that $f(S) =$ (V) . Then $U \cap f(S) =$ (VI) . Therefore $f^{-1}(U \cap f(S)) =$ (VII) .
Now we have (VIII) $\in f^{-1}(U \cap f(S))$ and (IX) . Then by the definition of subset relation,
(X) .

4. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be the function defined by $f(z) = \frac{2z|z|}{1 + |z|^2}$ for any $z \in \mathbb{C}$. Let $D = \{w \in \mathbb{C} : |w| < 2\}$.

Prove the statements below, with reference to the definition of *image sets*.

- (a) $f(\mathbb{C}) \subset D$.
- (b) $^\diamond$ $D \subset f(\mathbb{C})$.

5. $^\diamond$ Let $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be the function defined by $f(z) = \frac{i\bar{z}}{z}$ for any $z \in \mathbb{C} \setminus \{0\}$.

Let $H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$, and $S = \{w \in \mathbb{C} : |w| = 1\}$.

Prove the statements below, with reference to the definition of *image sets*.

- (a) $f(H) \subset S \setminus \{-i\}$.

- (b) $S \setminus \{-i\} \subset f(H)$.
6. (a)[◇] Is the statement (#) true? Justify your answer with reference to the definition of *pre-image sets*:
 (#) Let A, B be sets, and $f : A \rightarrow B$ be a function. Let U, V be subsets of B . Suppose $U \subset V$. Then $f^{-1}(U) \subset f^{-1}(V)$.
- (b)[◇] Is the statement (b) true? Justify your answer with reference to the definition of *pre-image sets*:
 (b) Let A, B be sets, and $f : A \rightarrow B$ be a function. Let U, V be subsets of B . Suppose $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.
7. (a)[◇] Is the statement (#) true? Justify your answer with reference to the definitions of *image sets* and *pre-image sets*:
 (#) Suppose A, B are sets, and $f : A \rightarrow B$ is a function. Then for any subset S of A , $S \subset f^{-1}(f(S))$.
- (b)[◇] Is the statement (b) true? Justify your answer with reference to the definitions of *image sets* and *pre-image sets*:
 (b) Suppose A, B are sets, and $f : A \rightarrow B$ is a function. Then for any subset S of A , $f^{-1}(f(S)) \subset S$.
8. (a) Fill in the blanks in the passage below so as to give the respective definitions for the notions of *relation in a set*, *reflexivity*, *symmetry*, *transitivity*, *equivalence relation*:—
- Suppose A, G are sets, and $R = (A, A, G)$.
 Then we say that R is a **relation in the set** A with graph (I) if (II) .
 - Suppose R is a relation in A with graph G . Then:—
 - * We say R is **reflexive** if the statement (ρ) holds:—
 (ρ) : (III) .
 - * We say R is **symmetric** if the statement (σ) holds:—
 (σ) : (IV) .
 - * We say R is **transitive** if the statement (τ) holds:—
 (τ) : (V) .
 - * We say R is an **equivalence relation in** A if (VI) .
- (b) You are not required to justify your answers in this part. There may be more than one correct answer in each sub-part, but you are only required to give one of them.
- i. Let $A = [0, +\infty)$ and G, H be the subsets of $\mathbb{R}^2$ defined respectively by

$$G = \{(x, x) \mid x > 0\}, \quad H = \{(x, y) \mid x \geq 0 \text{ and } y > 0 \text{ and } x^2 + y^2 = 1\}.$$

Name some appropriate ordered pairs $(s, t), (u, v) \in A^2$, if such exist, for which the ordered triple $(A, A, (G \cup H \cup \{(s, t), (u, v)\}))$ is a reflexive and symmetric relation in A .

ii. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and

$$\begin{aligned} A &= [0, +\infty), & B &= \{(x, y) \mid y = \alpha x\}, \\ C &= \{(x, y) \mid y = 2x + 1\}, & D &= \{(x, y) \mid \beta y = x + \gamma\}, \\ E &= \{(x, y) \mid (x - 1)^2 + (y - \delta)^2 = \delta^2\}, & G &= A^2 \cap (B \cup C \cup D \cup E). \end{aligned}$$

Suppose (A, A, G) is a reflexive and symmetric relation.

What are the values of $\alpha, \beta, \gamma, \delta$?

- iii. Let $A = \{0, 1, 2, 3\}$, and $G = \{(0, 1), (1, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$.
- A. Write down an ordered pair (s, t) for which $(A, A, G \cup \{(s, t)\})$ is a reflexive relation.
- B. Write down an ordered pair (u, v) for which $(A, A, G \setminus \{(u, v)\})$ is a symmetric relation.
- C. Fill in the blanks in the passage below, labelled by capital-letter Roman numerals, so as to obtain a valid justification for the statement ' (A, A, G) is not a transitive relation':
- [We verify the statement '(I) x, y, z (II) such that (III) $(x, z) \notin G$ ']
 Suppose
 $x = \text{(IV)}, y = \text{(V)}$ and $z = \text{(VI)}$.

Then $x, y, z \in A$.

For these same x, y, z , we have

$$(x, y) \in G \text{ and } (y, z) \underline{\hspace{1cm}} \text{ (VII) } \text{ and } (x, z) \underline{\hspace{1cm}} \text{ (VIII) }.$$

Hence (A, A, G) is not transitive (according to the definition of transitivity).

(c) You are not required to justify your answers in this part.

Denote by A the set of all non-zero polynomial functions on $\mathbb{R}$. Define

$$\begin{aligned} J &= \left\{ (f, g) \mid f, g \in A \text{ and } \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \text{ exists in } \mathbb{R} \right\}, \\ K &= \left\{ (f, g) \mid f, g \in A \text{ and } \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \text{ exists in } \mathbb{R} \text{ and is positive} \right\}, \\ L &= \left\{ (f, g) \mid f, g \in A \text{ and } \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \text{ exists in } \mathbb{R} \text{ and is negative} \right\}. \end{aligned}$$

Suppose R, S, T are relations in A with graph J, K, L respectively.

For each statement below, decide whether it is true or false:—

- | | | | | |
|-----------------------|-------------------------|------------------------|-------------------------|------------------------|
| i. R is reflexive. | iii. R is transitive. | v. S is symmetric. | vii. T is reflexive. | ix. T is transitive. |
| ii. R is symmetric. | iv. S is reflexive. | vi. S is transitive. | viii. T is symmetric. | |

9. Write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Define $E = \left\{ (\zeta, \eta) \in (\mathbb{C}^*)^2 : \frac{\eta}{\zeta} \in \mathbb{R}^* \right\}$, and $R = (\mathbb{C}^*, \mathbb{C}^*, E)$. Note that E is a subset of $(\mathbb{C}^*)^2$, and hence R is a relation in $\mathbb{C}^*$ with graph E .

Denote by $(M1)$, $(M2)$, $(M3)$ the statements below:—

- | | | |
|------------------------|-------------------------|--------------------------------------|
| (M1) R is reflexive. | (M2) R is transitive. | (M3) R is an equivalence relation. |
|------------------------|-------------------------|--------------------------------------|

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs of the statements $(M1)$, $(M2)$, $(M3)$ in the corresponding blocks below with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

(a) Here we prove the statement $(M1)$:—

We verify the statement ‘ (I) , $(\zeta, \zeta) \in E$ ’:

 (II) ζ (III) . We have (IV) $= 1 \in$ (V) . Then (ζ, ζ) (VI) .

It follows that R is reflexive.

(b) Here we prove the statement $(M2)$:—

We verify the statement ‘For any $\zeta, \eta, \xi \in \mathbb{C}^*$, (I) ’:

Pick any $\zeta, \eta, \xi \in \mathbb{C}^*$. (II)

Since $(\zeta, \eta) \in E$, we have (III) . Since (IV) , we have $\frac{\xi}{\eta} \in \mathbb{R}^*$.

Note that $\frac{\xi}{\zeta} = \frac{\xi}{\eta} \cdot \frac{\eta}{\zeta}$. Since $\frac{\eta}{\zeta} \in \mathbb{R}$ and $\frac{\xi}{\eta} \in \mathbb{R}$, we have (V) . Since (VI) , we have $\frac{\xi}{\zeta} \neq 0$.

Now we have $\frac{\xi}{\zeta} \in \mathbb{R}$ and $\frac{\xi}{\zeta} \neq 0$. Then (VII) . Hence (VIII) $\in E$.

It follows that R is transitive.

(c) Here we prove the statement $(M3)$:—

[We have already verified that R is reflexive and transitive.]

We want to verify that R is (I) . We verify the statement ‘ (II) ’:

(III) . Suppose (IV) . Then $\frac{\eta}{\zeta} \in \mathbb{R}^*$.

Since (V) , $\frac{1}{(\eta/\zeta)}$ is well-defined as a non-zero number, and $\frac{\zeta}{\eta} = \frac{1}{(\eta/\zeta)}$.

Since $\frac{\eta}{\zeta} \in \mathbb{R}$, we have (VI) .

We have $\frac{\zeta}{\eta} \in \mathbb{R}$ and $\frac{\zeta}{\eta} \neq 0$. Then $\frac{\zeta}{\eta} \in \mathbb{R}^*$. Therefore (VII) .

It follows that R is symmetric.

Since R is (VIII) , R is an equivalence relation.

10. Let $E = \{(x, y) \in \mathbb{R}^2 : x - y = a \text{ for some } a \in \mathbb{Q}\}$. Define $S = (\mathbb{R}, \mathbb{R}, E)$.

Denote by (N) the statement below:—

(N) S is an equivalence relation in $\mathbb{R}$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proof of the statement (N) in the block below with appropriate words/symbols so as to obtain a *complete* proof for the statement (N).

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

By definition, E is (I) . Then S is a relation in $\mathbb{R}$.

- [We want to verify the reflexivity of S .

We verify the statement ‘ (II) ’]

(III) .

Note that (IV) , and $0 \in \mathbb{Q}$. Then (V) , by the definition of S .

It follows that S is reflexive.

- [We want to verify the symmetry of S .

We verify the statement ‘ (VI) ’]

Pick any $x, y \in \mathbb{R}$. (VII) .

Then (VIII) such that $x - y = a$.

Note that (IX) .

Since (X) , we have $-a \in \mathbb{Q}$.

Now we have $y - x = -a$ and (XI) . Then (XII) , by the definition of S .

It follows that S is symmetric.

- [We want to verify the transitivity of S .

We verify the statement ‘ (XIII) ’]

Pick any (XIV) . Suppose (XV) .

Since $(x, y) \in E$, there exists some $a \in \mathbb{Q}$ such that (XVI) .

(XVII) .

Note that $x - z = (x - y) + (y - z) = a + b$.

(XVIII) .

Now we have $x - z = a + b$, and $a + b \in \mathbb{Q}$. Then (XIX) , by the definition of S .

It follows that S is transitive.

Since S is reflexive, symmetric and transitive, S is an equivalence relation in $\mathbb{R}$.

11. Define $G = \{(x, y) \in \mathbb{R}^2 : x - y > xy\}$, and $S = (\mathbb{R}, \mathbb{R}, G)$. Note that G is a subset of $\mathbb{R}^2$, and hence S is a relation in $\mathbb{R}$.

Denote by (O1), (O2), (O3) the statements below:—

- (O1) S is reflexive. (O2) S is symmetric. (O3) S is transitive.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* dis-proofs against the statements (O1), (O2), (O3) in the corresponding blocks below with appropriate words/symbols so as to obtain a *complete* dis-proof against each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

- (a) Here we dis-prove the statement (O1):—

We verify the statement ‘ _____ (I) _____ such that _____ (II) _____ ’:

_____ (III) _____. Note that $x_0 \in \mathbb{R}$.

Also note that _____ (IV) _____ = $0 \leq 0$ = _____ (V) _____. Therefore ‘ $x_0 - x_0 > x_0 \cdot x_0$ ’ is false.

Hence, by the definition of G , _____ (VI) _____.

It follows that S is not reflexive.

- (b) Here we dis-prove the statement (O2):—

We verify the statement ‘ _____ (I) _____ such that _____ (II) _____ ’:

Take _____ (III) _____. Note that $x_0, y_0 \in \mathbb{R}$.

Note that $x_0 - y_0 =$ _____ (IV) _____ = $x_0 y_0$. Then by the definition of G , _____ (V) _____.

Also note that $y_0 - x_0 = -1 \leq 0 = y_0 x_0$. Then ‘ _____ (VI) _____ ’ is false.

Therefore by the definition of G , _____ (VII) _____.

Hence for the same $x_0, y_0 \in \mathbb{R}$, _____ (VIII) _____ simultaneously.

It follows that S is not symmetric.

- (c) Here we dis-prove the statement (O3):—

We verify the statement ‘ _____ (I) _____ such that _____ (II) _____ ’:

Take $x_0 = -2, y_0 = 3$, _____ (III) _____. Note that $x_0, y_0, z_0 \in \mathbb{R}$.

Note that _____ (IV) _____. Then _____ (V) _____, $(x_0, y_0) \in G$.

Also note that $y_0 - z_0 = 5 > -6 = y_0 z_0$. Then by the definition of G , _____ (VI) _____.

Finally, note that _____ (VII) _____. Then ‘ $x_0 - z_0 > x_0 z_0$ ’ is false.

Therefore by the definition of G , _____ (VIII) _____.

Hence for the same $x_0, y_0, z_0 \in \mathbb{R}$, _____ (IX) _____ simultaneously.

It follows that S is not transitive.

12. Define the relation $S = (\mathbb{C}, \mathbb{C}, F)$ in $\mathbb{C}$ by $F = \{(\zeta, \xi) \in \mathbb{C}^2 : \zeta^2 - \xi^2 = ai \text{ for some } a \in \mathbb{R}\}$.

- (a) Verify that S is reflexive.
(b) Verify that S is symmetric.
(c) Verify that S is an equivalence relation in $\mathbb{C}$.

13. Let $R = (\mathbb{R}, \mathbb{R}, G)$ be the relation in $\mathbb{R}$ defined by $G = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } |y - x| \leq 1\}$.

- (a) Verify that R is reflexive.
(b) Is R symmetric? Justify your answer.
(c) Is R transitive? Justify your answer.

(d) Is R an equivalence relation in $\mathbb{R}$? Why?

14. Let $A = \left\{ \varphi \mid \begin{array}{l} \varphi : \mathbb{N} \longrightarrow \mathbb{R} \text{ is a function} \\ \text{and } \varphi(0) = 0. \end{array} \right\}$.

Define the relation $R = (A, A, H)$ in A by $H = \left\{ (\varphi, \psi) \in A^2 : \begin{array}{l} \text{For any } n \in \mathbb{N}, \\ \varphi(n+1) - \varphi(n) \leq \psi(n+1) - \psi(n) \end{array} \right\}$.

- (a) Verify that R is reflexive.
- (b) Verify that R is transitive.
- (c) Is R an equivalence relation in A ? Justify your answer.

15. Define the relation $R = (\mathbb{Z}, \mathbb{Z}, G)$ in $\mathbb{Z}$ by $G = \left\{ (x, y) \in \mathbb{Z}^2 : \begin{array}{l} \text{There exist some } m, n \in \mathbb{N} \\ \text{such that } y = mx + n. \end{array} \right\}$.

- (a) Verify that R is reflexive.
- (b) Verify that R is transitive.
- (c) Is R an equivalence relation in $\mathbb{Z}$? Justify your answer.

16. ♣ Let A be the set of all real-valued functions with domain $[1, +\infty)$.

Let R be the relation in A with graph E given by

$$E = \left\{ (f, g) \mid \begin{array}{l} f \in A \text{ and } g \in A \\ \text{and (there exist some positive real numbers } \alpha, K \\ \text{such that } \lim_{t \rightarrow +\infty} \frac{f(t) - Kg(t)}{t^\alpha} \text{ exists and equals 0).} \end{array} \right\}.$$

To answer the questions below, you may take for granted any standard results concerned with limits of functions in basic calculus of one real variable.

- (a) Is R reflexive? Justify your answer.
- (b) Is R symmetric? Justify your answer.
- (c) Is R an equivalence relation in A ? Justify your answer.

17. (a) Fill in the blanks in the passage below so as to give the definition for the notions of *equipotence*, *subpotence* and *strict subpotence*:—

Suppose A, B are sets. Then:—

- We say that A is **equipotent to** B if (I) _____ relation f from A to B such that _____ (II) _____.
- We say that A is **subpotent to** B if (III) _____ relation f from A to B such that _____ (IV) _____.
- We say that A is **strictly subpotent to** B if _____ (V) _____ and _____ (VI) _____.

(b) Explain the words/phrases below by providing appropriate definitions:—

- i. infinite sets.
- ii. countable sets.

(c) State, without proof, the results below:—

- i. Schröder-Bernstein Theorem.
- ii. Cantor's Theorem (on the power set of any given set).

(d) You are not required to justify your answers in this part.

- i. Consider the sets below. Classify them according to whether such a set is equipotent to $\mathbb{N}$, or whether it is equipotent to $\mathfrak{P}(\mathbb{N})$, or whether it is of equipotent to $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$.

$\mathbb{N}$,	$\mathbb{Q}$,	$\mathbb{R} \setminus \mathbb{Q}$,	$\text{Map}(\{0, 1\}, \mathbb{N})$,
$\mathbb{N}^3$,	$[0, 1]$,	$\mathfrak{P}(\mathbb{N})$,	$\text{Map}(\mathbb{N}, \{0, 1\})$,
$\mathbb{N} \times \mathbb{R}$,	$\mathbb{C}$,	$\mathfrak{P}(\mathbb{R})$,	$\text{Map}(\mathbb{R}, \{0, 1\})$.

ii. For each statement below, decide whether it is true or false:—

- A. $\mathbb{N} \sim \{x \in \mathbb{N} : x \geq 1050\}$.
- B. $[1010, 1030] \lesssim (2040, 2050)$.
- C. $\mathbb{R}^2 < \mathbb{R}^3$.
- D. $\mathfrak{P}(\mathbb{N}) \lesssim \mathbb{Z} \times \mathbb{Q}^4$.

E. $\text{Map}(\mathbb{N}, \{0, 1\}) \lesssim \mathfrak{P}(\mathbb{R})$.

18. (a) Fill in the blanks in the passage below so as to give the statement of the *Glueing Lemma*:—

Let C, C', D, D' be sets, g be a relation from C to D with graph G , and g' be a relation from C' to D' with graph G' .

Suppose (I) $\quad = \emptyset$ and $D \cap D' =$ (II) $\quad$ and (III) $\quad$.

Then (IV) $\quad$.

(b) Let $J = [0, 1)$, $K = (0, 1]$.

i. Denote by (P) the statement below:— (P) J is equipotent to K .

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proof of the statement (P) in the block below with appropriate words/symbols so as to obtain a *complete* proof for the statement (P) . (The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

Suppose $H = \{(x, y) \mid x \in J \text{ and } y \in K \text{ and } x + y = 1\}$, and $h = (J, K, H)$.

Note that (I) $\quad$. Hence h is a relation from J to K with graph H .

We verify the statement (E) : ‘for any $x \in J$, (II) $\quad$ ’.

Pick any (III) $\quad$. Take (IV) $\quad$. By definition, we have $x + y = x + (1 - x) = 1$.

Since (V) $\quad$, we have $0 \leq x < 1$.

Since $x \geq 0$, we have (VI) $\quad$. Since $x < 1$, we have (VII) $\quad$.

Then $0 < y \leq 1$. Therefore (VIII) $\quad$.

Hence (IX) $\quad$.

We verify the statement (U) : ‘for any $x \in J$, for any $y, z \in K$, (X) $\quad$ ’.

(XI) $\quad$. Suppose (XII) $\quad$.

(XIII) $\quad$, we have $x + y = 1$. Then $y = 1 - x$.

Since $(x, z) \in H$, we have (XIV) $\quad$. Then (XV) $\quad$.

Therefore (XVI) $\quad$.

We verify the statement (S) : ‘(XVII) $\quad$ such that (XVIII) $\quad$ ’.

(XIX) $\quad$. Take $x = 1 - y$. By definition, we have $x + y = (1 - y) + y = 1$.

Since $y \in K$, we have (XX) $\quad$.

Since (XXI) $\quad$, we have $x = 1 - y < 1 - 0 = 1$. Since $y \leq 1$, we have (XXII) $\quad$.

Then $0 \leq x < 1$. Therefore $x \in J$.

Hence (XXIII) $\quad$.

We verify the statement (I) : ‘(XXIV) $\quad$ if $(x, y) \in H$ and $(w, y) \in H$ (XXV) $\quad$ ’.

Pick any $x, w \in J$, $y \in K$. (XXVI) $\quad$.

(XXVII) $\quad$, we have $x + y = 1$. Then $x = 1 - y$.

Since $(w, y) \in H$, we have (XXVIII) $\quad$. Then $w = 1 - y$.

Therefore (XXIX) $\quad$.

By (E) , (U) , (XXX) $\quad$. By (S) , (XXXI) $\quad$. By (I) , (XXXII) $\quad$.

Hence h is a bijective function from J to K with graph H .

It follows that (XXXIII) $\quad$.

ii. Let $L = (0, 1)$.

By applying the Glueing Lemma to ‘glue together’ the identity function id_L and some appropriate bijective function, prove that J is equipotent to K .

19. Let $J = [0, 1)$, $L = (0, 1)$, $M = [0, +\infty)$, $N = (0, +\infty)$.

- (a) By writing down appropriate bijective functions, verify that $J \sim M$ and $L \sim N$. Justify your answer.
- (b) By writing down an appropriate bijective function, verify that $L \sim \mathbb{R}$. Justify your answer.
20. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $H = \{w \in \mathbb{C} : \operatorname{Im}(w) > 0\}$.
- Define $F = \left\{ (z, w) \mid z \in D \text{ and } w \in H \text{ and } w = \frac{z+i}{iz+1} \right\}$, and $f = (D, H, F)$.
- (a) Is f a function? Justify your answer.
- (b) Is it true that $D \sim H$? Justify your answer, with direct reference to the definition of equipotence.
21. Let $I = (0, +\infty)$, $J = [-1, 1]$.
- (a) Prove that $\frac{1}{a+1} \in J$ for any $a \in I$.
- (b) Define the function $g : I \rightarrow J$ by $g(x) = \frac{1}{x+1}$ for any $x \in I$. Is g injective? Justify your answer.
- (c) Apply the Schröder-Bernstein Theorem to prove that $I \sim J$.
22. Let $A = [-1, 1]$, $B = (-4, -2] \cup [2, 4)$.
- (a) Name one injective function from A to B , if there is any at all, and verify that it is indeed an injective function from A to B .
- (b) Apply the Schröder-Bernstein Theorem, or otherwise, to prove that $A \sim B$.
23. $\diamond$ Let $A = [1010, 1050] \setminus \{1030\}$ and $B = (2040, 2050) \cup ([2060, +\infty) \cap \mathbb{Q})$.
- (a) Name one injective function from A to B , if there is any at all, and verify that it is indeed an injective function from A to B .
- (b) Apply the Schröder-Bernstein Theorem to prove that $A \sim B$.
24. Let $D = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. Define $F = \left\{ (z, w) \mid z \in \mathbb{C} \text{ and } w \in D \text{ and } w = \frac{iz|z|}{1+|z|+|z|^2} \right\}$. Note that $F \subset \mathbb{C} \times D$.
- Define $f = (\mathbb{C}, D, F)$.
- (a) $\diamond$ Verify that f is a function.
- (b) $\diamond$ Is f injective? Justify your answer.
- (c) Is f surjective? Justify your answer.
- (d) Is it true that $D \sim \mathbb{C}$? Justify your answer.

Remark. Where appropriate and relevant, you may apply the Schröder-Bernstein Theorem in your argument.