MATH1050BC/1058 Assignment 5 (Answers and selected solution)

1. (a) **Answer.**

   (I) for any real numbers a, a', b, b'

   (II) $z = a'\zeta + b'\overline{\zeta}$

   (III) a

   (IV) b

   (b) **Solution.**

   Let ζ be a complex number. Suppose ζ^2 is not a real number.

   Pick any complex number z. Pick any real numbers a, b, a', b'. Suppose $z = a\zeta + b\overline{\zeta}$ and $z = a'\zeta + b'\overline{\zeta}$.

   Then $a\zeta + b\overline{\zeta} = a'\zeta + b'\overline{\zeta}$.

   Therefore $(a - a')\zeta = (b' - b)\overline{\zeta}$.

   Hence $(a - a')\zeta^2 = (b' - b)|\zeta|^2$.

   Note that $b, b', |\zeta|^2$ are real numbers. Then $(b' - b)|\zeta|^2$ is a real number.

   Since a, a' are real number, $a - a'$ is also a real number. Then $(a - a')\mathsf{Im}(\zeta^2) = \mathsf{Im}((a - a')\zeta^2) = \mathsf{Im}((b' - b)|\zeta|^2) = 0$.

   By assumption, ζ^2 is not a real number. Then $\mathsf{Im}(\zeta^2) \neq 0$. Therefore $a - a' = 0$. Hence $a = a'$.

   Then $(b' - b)|\zeta|^2 = (a - a')\zeta^2 = 0$.

   By assumption, $\zeta \neq 0$. Then $b' - b = 0$. Therefore $b = b'$.

2. **Solution.**

   [We want to prove this statement: '*Let I be an interval in $\mathbb{R}$, and $f, g : I \longrightarrow \mathbb{R}$ be functions. Suppose f is strictly increasing on I and g is strictly decreasing on I.*
   Let $c, c' \in I$. Suppose $f(c) = g(c)$ and $f(c') = g(c')$. Then $c = c'$.]

   Let I be an interval in $\mathbb{R}$, and $f, g : I \longrightarrow \mathbb{R}$ be functions. Suppose f is strictly increasing on I and g is strictly decreasing on I.

   Pick any $c, c' \in I$. Suppose $f(c) = g(c)$ and $f(c') = g(c')$. We verify that $c = c'$ by the proof-by-contradiction method:—

   - Suppose it were true that $c \neq c'$.
     Without loss of generality, assume $c < c'$.
     Since f is strictly increasing on I, we would have $f(c) < f(c')$.
     Since g is strictly decreasing on I we would have $g(c) > g(c')$.
     Recall that $f(c) = g(c)$ and $f(c') = g(c')$.
     Then $f(c) < f(c') = g(c') < g(c) = f(c)$. Therefore $f(c) < f(c)$. Contradiction arises.
     Hence $c = c'$ in the first place.

3. **Solution.**

   (a) (M) is formally formulated as:—

   (M): For any set A, for any functions $f, g : A \longrightarrow A$, the equality $g \circ f = f \circ g$ as functions holds.

   Hence $(\sim M)$ reads:—

   $(\sim M)$: There exist some set A, and some functions $f, g : A \longrightarrow A$ such that $g \circ f \neq f \circ g$ as functions.

   (b) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = \dfrac{x^2}{1 + x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

   i. For any $x \in \mathbb{R}$, we have

   $$\begin{aligned} (g \circ f)(x) &= -\frac{1}{1 + x^2}, \\ (f \circ g)(x) &= \frac{(x - 1)^2}{1 + (x - 1)^2}. \end{aligned}$$

   ii. We have $(g \circ f)(0) = -1$ and $(f \circ g)(0) = \dfrac{1}{2}$. Hence $(g \circ f)(0) \neq (f \circ g)(0)$.

   iii. There exists some $x_0 \in \mathbb{R}$, namely, $x_0 = 0$, such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence it is not true that $g \circ f = f \circ g$ as functions.

(c) Let $A = \{0, 1\}$.

Define $f, g : A \longrightarrow A$ by $f(0) = f(1) = 0$, $g(0) = g(1) = 1$.

By definition, the function $g \circ f : A \longrightarrow A$ is given by $(g \circ f)(0) = g(f(0)) = g(0) = 1$, $(g \circ f)(1) = g(f(1)) = g(0) = 1$.

The function $f \circ g : A \longrightarrow A$ is given by $(f \circ g)(0) = f(g(0)) = f(1) = 0$, $(f \circ g)(1) = f(g(1)) = f(1) = 0$.

Take $x_0 = 0$. We have $(g \circ f)(x_0) = 1$ and $(f \circ g)(x_0) = 0$. Then $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

Therefore $g \circ f \neq f \circ g$ as functions.

Remark. Let A be a set. (This set is fixed in our subsequent discussion.) Suppose $f, g : A \longrightarrow A$ are two functions from the set A to A itself.

- When we want to verify that $g \circ f$, $f \circ g$ are the same function from A to A, we have to verify that for any $x \in A$, $(g \circ f)(x) = (f \circ g)(x)$.
- To verify that $g \circ f$, $f \circ g$ are not the same function from A to A, we check that there exists some $x_0 \in A$ such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence we have to name an appropriate x_0 and show that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

4. **Solution.**

(a) Suppose $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $\alpha(x) = \dfrac{2x}{x^2+1}$ for any $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$, we have $(X(\alpha))(x) = \dfrac{x^2}{x^2+1}$.

For any $x \in \mathbb{R}$, we have $(D(\alpha))(x) = \dfrac{-x^2+1}{(x^2+1)^2}$

For any $x \in \mathbb{R}$, we have $(I_0(\alpha))(x) = \dfrac{1}{2}\ln(x^2+1)$.

Hence $X(\alpha)$, $D(\alpha)$, $I_0(\alpha)$ are respectively the real-valued functions of one real variable, with domain $\mathbb{R}$, given by

$$(X(\alpha))(x) = \frac{x^2}{x^2+1}, \qquad (D(\alpha))(x) = \frac{-x^2+1}{x^2+1}, \qquad (I_0(\alpha))(x) = \frac{1}{2}\ln(x^2+1) \qquad \text{for any } x \in \mathbb{R}.$$

(b) i. Pick any $\varphi \in C^\infty(\mathbb{R})$, $x \in \mathbb{R}$. We have

$$\begin{aligned} ((D \circ X)(\varphi))(x) &= (D(X(\varphi)))(x) = (X(\varphi))'(x) = \frac{d}{dt}(t\varphi(t))\bigg|_{t=x} = x\varphi'(x) + \varphi(x), \\ (X \circ D)(\varphi))(x) &= X(D(\varphi))(x) = x(D(\varphi))(x) = x\varphi'(x) \end{aligned}$$

Then $((D \circ X)(\varphi))(x) - (X \circ D)(\varphi))(x) = (x\varphi'(x) + \varphi(x)) - x\varphi'(x) = \varphi(x)$.

ii. Pick any $\varphi \in C^\infty(\mathbb{R})$, $x \in \mathbb{R}$. We have

$$\begin{aligned} ((I_0 \circ X)(\varphi))(x) &= (I_0(X(\varphi)))(x) = \int_0^x (X(\varphi))(t)dt = \int_0^x t\varphi(t)dt \\ &= \int_0^x t(I_0(\varphi))'(t)dt \\ &= t(I_0(\varphi))(t)\bigg|_{t=0}^{t=x} - \int_0^x (I_0(\varphi))(t)dt \\ &= x(I_0(\varphi))(x) - 0 - (I_0(I_0(\varphi)))(x) \\ &= (X(I_0(\varphi)))(x) - (I_0(I_0(\varphi)))(x) = (X \circ I_0)(\varphi)(x) - (I_0 \circ I_0)(\varphi)(x) \end{aligned}$$

Then $((X \circ I_0)(\varphi))(x) - (I_0 \circ X)(\varphi))(x) = ((I_0 \circ I_0)(\varphi))(x)$.

5. **Answer.**

(a) i. Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then we say f is surjective if the statement (S) holds:—

(S) For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.

ii. Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then we say f is injective if the statement (I) holds:—

(I) For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.

(b) i. (I) For any

(II) there exists

(III) $y = f(x)$

(IV) Pick any

Alternative answer. Let

Alternative answer. Suppose

Alternative answer. Assume

Alternative answer. Take any

(V) $x = (y+1)^{\frac{3}{5}}$

(VI) $f(x) = x^{\frac{5}{3}} - 1 = \left[(y+1)^{\frac{3}{5}}\right]^{\frac{5}{3}} - 1 = (y+1) - 1 = y$

(VII) f is surjective.

ii. (I) For any

(II) if $f(x) = f(w)$

(III) $x, w \in \mathbb{R}$

(IV) Suppose

Alternative answer. Assume

(V) $f(x) + 1 = f(w) + 1$

(VI) $(w^{\frac{5}{3}})^{\frac{3}{5}} = w$

(VII) f is injective

(c) i. (I) There exists some

(II) for any

(III) $y_0 \neq f(x_0)$

(IV) Take

Alternative answer. Let

Alternative answer. Define

Alternative answer. Pick

Alternative answer. Suppose

Alternative answer. Assume

(V) for any $x \in \mathbb{R}$

(VI) Suppose

Alternative answer. Assume

(VII) there existed some $x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$.

(VIII) 0

(IX) $\left(x_0 - \frac{1}{2}\right)^2 + \frac{3}{4} \geq 0 + \frac{3}{4}$

(X) f is not surjective.

ii. (I) there exist some

(II) $= f(w_0)$

(III) $x_0 \neq w_0$

(IV) $w_0 = 2$.

(V) $x_0 \neq w_0$.

(VI) $f(w_0) = \dfrac{2}{2^2+1} = \dfrac{2}{5}$

(VII) $f(w_0)$

(VIII) f is not injective

6. **Answer.**

(a) (I) for any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $\zeta = f(z)$

(II) Pick any $\zeta \in \mathbb{C}$.

(III) there exists some $\theta \in \mathbb{R}$

(IV) Take $z = \sqrt[5]{|\zeta|} \cdot \left(\cos\left(\frac{\theta}{5}\right) + i\sin\left(\frac{\theta}{5}\right)\right)$.

(V) $\left[\sqrt[5]{|\zeta|} \cdot \left(\cos\left(\frac{\theta}{5}\right) + i\sin\left(\frac{\theta}{5}\right)\right)\right]^5 = \left(\sqrt[5]{|\zeta|}\right)^5 \cdot \left(\cos\left(5 \cdot \frac{\theta}{5}\right) + i\sin\left(5 \cdot \frac{\theta}{5}\right)\right) = |\zeta|(\cos(\theta) + i\sin(\theta)) = \zeta$

(VI) f is surjective

(b) (I) there exist some $z_0, w_0 \in \mathbb{C}$ such that $f(z_0) = f(w_0)$ and $z_0 \neq w_0$

(II) Take $z_0 = 1$, $w_0 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$.

(III) $z_0 \neq w_0$

(IV) $f(z_0) = {z_0}^5 = 1^5 = 1$

(V) $f(w_0) = {w_0}^5 = \cos\left(5 \cdot \frac{2\pi}{5}\right) + i\sin\left(5 \cdot \frac{2\pi}{5}\right) = \cos(2\pi) + i\sin(2\pi) = 1$

Alternative answer.

(IV) $f(w_0) = {w_0}^5 = \cos\left(5 \cdot \frac{2\pi}{5}\right) + i\sin\left(5 \cdot \frac{2\pi}{5}\right) = \cos(2\pi) + i\sin(2\pi) = 1$ (V) $f(z_0) = {z_0}^5 = 1^5 = 1$

(VI) f is not injective

7. (a) **Answer.**

i. Suppose A, B, C are sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ are functions. Define the function $g \circ f : A \longrightarrow C$ by $(g \circ f)(x) = g(f(x))$ for any $x \in A$. Then $g \circ f$ is called the composition of f, g.

ii. Suppose C is a set. Define the function $\mathsf{id}_C : C \longrightarrow C$ by $\mathsf{id}_C(x) = x$ for any $x \in C$. Then id_C is called the identity function on C.

(b) **Answer.**

(I) Suppose f is injective or f is surjective.
(II) $x \in A$
(III) $f(x)$
(IV) $(f \circ f)(x)$
(V) the definition of injectivity
(VI) $f(x) = x = \mathsf{id}_A(x)$
(VII) Suppose f is surjective.
(VIII) Pick any
(IX) there exists some $u \in A$ such that $x = f(u)$
(X) $(f \circ f)(u)$
(XI) $(f \circ f)(u) = f(u)$
(XII) $f = \mathsf{id}_A$

(c) **Solution.**
Let B be a set, K be a subset of B, and $\varphi : \mathfrak{P}(B) \longrightarrow \mathfrak{P}(B)$ be the function defined by $\varphi(S) = S \cap K$ for any $S \in \mathfrak{P}(B)$. Suppose φ is injective or φ is surjective.
We verify that $\varphi \circ \varphi = \varphi$:

- Pick any $S \in \mathfrak{P}(B)$. Then $\varphi(S) = S \cap K = S \cap (K \cap K) = (S \cap K) \cap K = \varphi(S \cap K) = \varphi(\varphi(S)) = (\varphi \circ \varphi)(S)$. It follows that $\varphi \circ \varphi = \varphi$.

By the result described in the Statement (T), we have $\varphi = \mathsf{id}_{\mathfrak{P}(B)}$.
Since K is a subset of B, we have $K = B \cap K$. Then $K = B \cap K = \varphi(B) = \mathsf{id}_{\mathfrak{P}(B)}(B) = B$.

8. **Answer.**

(a) (I) Pick any $x \in A$
(II) Pick any $y \in B$
(III) $f(t) = y$ for any $t \in A$
There are many correct answers: as long as the 'formula of f' is 'complete' and $f(x) = y$ according to the 'formula of f'.
(IV) $\mathsf{Map}(A, B)$
(V) $E_x(f) = f(x) = y$

(b) (I) some distinct
(II) there exists some $u \in A$ such that E_u is injective
(III) $x \in A \backslash \{u\}$
(IV) Define $f : A \longrightarrow B$ by
(V) for any
(VI) $g : A \longrightarrow B$
(VII) $t = u$
(VIII) $f(u) = y = g(u) = E_u(g)$
(IX) $f = g$
(X) $y \neq z$
(XI) $f \neq g$
(XII) and

9. **Answer.**

(a) Suppose A, B are sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ are functions.
Then we say g is an inverse function of f if $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$.
Alternative answer.
Suppose A, B are sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ are functions.
Then we say g is an inverse function of f if both statements $(\star)$, $(\star\star)$ hold:—

$(\star)$ For any $x \in A$, $(g \circ f)(x) = x$.
$(\star\star)$ For any $y \in B$, $(f \circ g)(y) = y$.

(b) i. $a = -9$, $b = 16$.

ii. The inverse function of f, which is the function $f^{-1} : [-9, 16] \longrightarrow [-2, 3]$, is given by $f^{-1}(y) = \sqrt{4 + \sqrt{y + 9}}$ for any $y \in [-9, 16]$.

iii. A. $\sqrt{5}$
B. π

10. **Answer.**

(a) $J = (1, +\infty)$.

(b) The inverse function of f, which is the function $f^{-1} : J \longrightarrow (0, +\infty)$, is given by $f^{-1}(y) = \dfrac{1}{4}\left(\ln\left(\dfrac{y+1}{y-1}\right)\right)^2$ for any $y \in J$.

11. **Solution.**

Denote the interval $(0, +\infty)$ by I. Let $f : I \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ for any $x \in I$.

(a) Pick any $x, w \in I$. Suppose $f(x) = f(w)$. Then $\dfrac{1}{2}\left(x - \dfrac{1}{x}\right) = \dfrac{1}{2}\left(w - \dfrac{1}{w}\right)$.

We have $x^2w - w = w^2x - x$.

Therefore $xw(x - w) = x^2w - w^2x = w - x$. Hence $(xw + 1)(x - w) = 0$.

Since $x, w \in I$, we have $x > 0, w > 0$ and $xw + 1 > 0$. Therefore $x - w = 0$. Hence $x = w$.

It follows that f is injective.

(b) Pick any $y \in \mathbb{R}$. Take $x = y + \sqrt{y^2 + 1}$. Note that $x \in I$.

We have

$$\begin{aligned} f(x) &= \frac{1}{2}\left(x - \frac{1}{x}\right) \\ &= \frac{1}{2}\left(y + \sqrt{y^2+1} - \frac{1}{y + \sqrt{y^2+1}}\right) \\ &= \frac{1}{2}\left[y + \sqrt{y^2+1} - \frac{y - \sqrt{y^2+1}}{\left(y + \sqrt{y^2+1}\right)\left(y - \sqrt{y^2+1}\right)}\right] = \frac{1}{2}\left(y + \sqrt{y^2+1} + y - \sqrt{y^2+1}\right) = y \end{aligned}$$

It follows that f is surjective.

(c) The inverse function of f, which is the function $f^{-1} : \mathbb{R} \longrightarrow I$, is given by $f^{-1}(y) = y + \sqrt{y^2 + 1}$ for any $y \in \mathbb{R}$.

12. (a) i. **Solution.**

Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $h(z) = \bar{z}$ for any $z \in \mathbb{C}$.

- [We verify that h is injective. This amounts to verifying: ‘ *For any* $z, w \in \mathbb{C}$, *if* $h(z) = h(w)$ *then* $z = w$.’]
  Let $z, w \in \mathbb{C}$. Suppose $h(z) = h(w)$. Then $\bar{z} = \bar{w}$. Therefore $z = \bar{\bar{z}} = \bar{\bar{w}} = w$.
  It follows that h is injective.
- [We verify that h is surjective. This amounts to verifying: ‘ *For any* $\zeta \in \mathbb{C}$, *there exists some* $z \in \mathbb{C}$ *such that* $h(z) = \zeta$.’]
  Let $\zeta \in \mathbb{C}$. Take $z = \bar{\zeta}$. By definition, $z \in \mathbb{C}$. Also by definition, $h(z) = \bar{z} = \bar{\bar{\zeta}} = \zeta$.
  It follows that h is surjective.

Hence h is bijective.

ii. **Answer.**

The inverse function $h^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function h is given by $h^{-1}(z) = \bar{z}$ for any $z \in \mathbb{C}$.

(b) **Solution.**

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2\bar{z}$ for any $z \in \mathbb{C}$.

i. [We want to verify the statement ‘*for any* $w \in \mathbb{C}$, *there exists some* $z \in \mathbb{C}$ *such that* $f(z) = w$.’]

Pick any $w \in \mathbb{C}$. We have $w = 0$ or $w \neq 0$.

* (Case 1.) Suppose $w = 0$. Then $f(0) = 0 = w$.
* (Case 2.) Suppose $w \neq 0$. (Note that $|w| \neq 0$. Then $\dfrac{1}{\sqrt[3]{|w|}}$ is well-defined as a complex number.)

  Define $z = \dfrac{w}{(\sqrt[3]{|w|})^2}$. By definition, $z \in \mathbb{C}$.

  We have $f(z) = z^2\bar{z} = \left(\dfrac{w}{(\sqrt[3]{|w|})^2}\right)^2 \overline{\left[\dfrac{w}{(\sqrt[3]{|w|})^2}\right]} = \dfrac{w^2\bar{w}}{(\sqrt[3]{|w|})^6} = \dfrac{w|w|^2}{|w|^2} = w$.

It follows that f is surjective.

ii. [We want to verify the statement ‘*for any* $u, v \in \mathbb{C}$, *if* $f(u) = f(v)$ *then* $u = v$.’]

Pick any $u, v \in \mathbb{C}$. Suppose $f(u) = f(v)$.

Then $|u|^3 = |u^2\bar{u}| = |f(u)| = |f(v)| = |v^2\bar{v}| = |v|^3$.

Since $|u|, |v| \in \mathbb{R}$, we have $|u| = |v|$.

Now $u|u|^2 = u^2\bar{u} = f(u) = f(v) = v^2\bar{v} = v|v|^2 = v|u|^2$.

Then $(u - v)|u|^2 = 0$. Therefore $u - v = 0$ or $|u|^2 = 0$.

* (Csae 1.) Suppose $u - v = 0$. Then $u = v$.
* (Case 2.) Suppose $u - v \neq 0$. Then $|u|^2 = 0$. Therefore $u = 0$.
  Also, $|v| = |u| = 0$. Then $u = 0 = v$.

Hence, in any case $u = v$.
It follows that f is injective.

(c) **Solution.**
Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Let $g : \mathbb{C}^* \longrightarrow \mathbb{C}$ be the function defined by $g(z) = z/\bar{z}$ for any $z \in \mathbb{C}^*$.

i. [We want to verify the statement '*there exist some* $u_0, v_0 \in \mathbb{C}^*$ *such that* $g(u_0) = g(v_0)$ *and* $u_0 \neq v_0$'.]
Take $u_0 = 1$, $v_0 = -1$.
We have $u_0, v_0 \in \mathbb{C}^*$ and $u_0 \neq v_0$.
We have $g(u_0) = 1$ and $g(v_0) = 1$. Then $g(u_0) = g(v_0)$.
It follows that g is not injective.

ii. [We want to verify the statement '*There exists some* $w_0 \in \mathbb{C}$ *such that for any* $z \in \mathbb{C}^*$, $g(z) \neq w_0$.']
Take $w_0 = 2$. Note that $w_0 \in \mathbb{C}$. We verify with the proof-by-contradiction method, that for any $z \in \mathbb{C}^*$, $g(z) \neq w_0$:

- Suppose there existed some $z \in \mathbb{C}^*$ such that $g(z) = w_0$.
Then $|g(z)| = |w_0| = 2$.
For the same z, we have $|g(z)| = |z/\bar{z}| = |z|/|\bar{z}| = |z|/|z| = 1 \neq 2$.
Now $|g(z)| = 2$ and $|g(z)| \neq 2$. Contradiction arises.

It follows that g is not surjective.

13. **Solution.**
Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

(a) Let $z \in \mathbb{C}$.
Note that $\dfrac{az+b}{cz+d} - \dfrac{a}{c} = \dfrac{c(az+b) - a(cz+d)}{c(cz+d)} = -\dfrac{ad-bc}{c(cz+d)} \neq 0$, because $ad - bc \neq 0$.
Then $\dfrac{az+b}{cz+d} \neq \dfrac{a}{c}$.

(b) Define the function $f : \mathbb{C}\backslash\{-d/c\} \longrightarrow \mathbb{C}\backslash\{a/c\}$ by $f(z) = \dfrac{az+b}{cz+d}$ for any $z \in \mathbb{C}\backslash\{-d/c\}$.

i. Pick any $z, w \in \mathbb{C}\backslash\{-d/c\}$. Suppose $f(z) = f(w)$.
Then $\dfrac{az+b}{cz+d} = \dfrac{aw+b}{cw+d}$. Therefore

$$aczw + bd + adz + bcw = (az+b)(cw+d) = (aw+b)(cz+d) = aczw + bd + adw + bcz.$$

Hence $(ad-bc)z = (ad-bc)w$. Since $ad - bc \neq 0$, we have $z = w$.
It follows that f is injective.

ii. Pick any $\zeta \in \mathbb{C}\backslash\{a/c\}$. Since $\zeta \neq \dfrac{a}{c}$, we have $-c\zeta + a \neq 0$. Take $z = \dfrac{d\zeta - b}{-c\zeta + a}$. By definition, $z \in \mathbb{C}$.
Moreover, $z - \dfrac{-d}{c} = \dfrac{d\zeta - b}{-c\zeta + a} + \dfrac{d}{c} = \dfrac{c(d\zeta - b) + d(-c\zeta + a)}{c(-c\zeta + a)} = \dfrac{ad - bc}{c(-c\zeta + a)} \neq 0$.
Then $z \neq -\dfrac{d}{c}$. Hence $z \in \mathbb{C}\backslash\{-d/c\}$.
Also by definition, $f(z) = \dfrac{a[(d\zeta - b)/(-c\zeta + a)] + b}{c[(d\zeta - b)/(-c\zeta + a)] + d} = \dfrac{a(d\zeta - b) + b(-c\zeta + a)}{c(d\zeta - b) + d(-c\zeta + a)} = \dfrac{(ad - bc)\zeta + 0}{0 \cdot \zeta + (ad - bc)} = \zeta$.
It follows that f is surjective.

iii. The inverse function of f, which is $f^{-1} : \mathbb{C}\backslash\{a/c\} \longrightarrow \mathbb{C}\backslash\{-d/c\}$, is given by $f^{-1}(\zeta) = \dfrac{d\zeta - b}{-c\zeta + a}$ for any $\zeta \in \mathbb{C}\backslash\{a/c\}$.

14. **Solution.**
Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} \cos\left(\dfrac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

(a) Take $x_0 = \dfrac{2^{2/3}}{\pi^{2/3}}$, $w_0 = \dfrac{2^{2/3}}{3^{2/3}\pi^{2/3}}$.
Note that $x_0, w_0 \in (0, +\infty)$ and $x_0 \neq w_0$.
Also note that $f(x_0) = 0 = f(w_0)$.
It follows that f is not injective.

(b) i. Let $x \in (0, +\infty)$.

By the Triangle Inequality for the reals, $|x^2 - 2x + 4| \leq |x^2 + 4| + |2x| = x^2 + 2x + 4 = |x^2 + 2x + 4|$.

Note that $|x^2 + 2x + 4| > 0$.

Then $\left| \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} \right| \leq 1$.

ii. Take $y_0 = 2$. We verify that for any $x \in (0, +\infty)$, we have $f(x) \neq y_0$:

- Let $x \in (0, +\infty)$.

  We have $|f(x)| = \left| \frac{x^2-2x+4}{x^2+2x+4} \cos\left(\frac{1}{x\sqrt{x}}\right) \right| \leq \left| \frac{x^2-2x+4}{x^2+2x+4} \right| \cdot \left| \cos\left(\frac{1}{x\sqrt{x}}\right) \right| \leq 1 \cdot 1 = 1 < 2$.

  Then $f(x) \neq 2$.

It follows that f is not surjective.

15. **Solution.**

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} x^9 & \text{if} \quad x \in \mathbb{Q} \\ -x^3 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

i. We verify that f is injective:

- Pick any $x, w \in \mathbb{R}$. Suppose $f(x) = f(w)$.

  We verify the statement '*x, w are both rational or x, w are both irrational*' with the method of proof-by-contradiction:

  Suppose it were true that one of x, w was rational and the other was irrational.
  Without loss of generality, assume x was rational and w was irrational.
  Then by the definition of f, we would have $f(x) = x^9$ and $f(w) = -w^3$.
  Therefore by assumption, $-w^3 = f(w) = f(x) = x^9$.
  Since $x, w \in \mathbb{R}$, we have $w = -x^3$. Then, since x was rational, w would be rational.
  But w was assumed to be irrational. Then w was simultaneously rational and irrational. Contradiction arises.
  It follows, in the first place, that x, w are both rational or x, w are both irrational.

  We now verify that $x = w$:

  * (Case 1.) Suppose x, w are both rational.
    Then by the definition of f, we have $x^9 = f(x) = f(w) = w^9$. Since $x, w \in \mathbb{R}$, we have $x = w$.
  * (Case 2.) Suppose x, w are both irrational. Then by the definition of f, we have $-x^3 = f(x) = f(w) = -w^3$. Then, since $x, w \in \mathbb{R}$, we have $x = w$.

  Hence in any case, $x = w$.
  It follows that f is injective.

ii. We verify that f is not surjective:

- Take $y_0 = 8$. Pick any $t \in \mathbb{R}$.
  We verify $y_0 \neq f(t)$ by the method of proof-by-contradiction.
  Suppose it were true that $y_0 = f(t)$.
  Note that t is rational or t is irrational.
  * (Case 1). Suppose t is rational.
    Then by the definition of f, we would have $8 = y_0 = f(t) = t^9$. Since $t \in \mathbb{R}$, we would have $t = \sqrt[3]{2}$. But $\sqrt[3]{2}$ is irrational.
    Contradiction arises.
  * (Case 2). Suppose t is irrational. Then by the definition of f, we would have $8 = y_0 = f(t) = -t^3$. Since $t \in \mathbb{R}$, we would have $t = -2$. But -2 is rational.
    Contradiction arises.

  In any case, contradiction arises. It follows that in the first place, $y_0 \neq f(t)$.
  Hence f is not surjective.

(b) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = \begin{cases} x^3 & \text{if} \quad x \in \mathbb{Q} \\ -x^9 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

i. We verify that g is not injective:

- Take $x_0 = 2$, $w_0 = -\sqrt[3]{2}$.
  Note that $x_0 \neq w_0$, $x_0 \in \mathbb{Q}$, and $w_0 \in \mathbb{R}\backslash\mathbb{Q}$.
  Since $x_0 \in \mathbb{Q}$, we have $g(x_0) = {x_0}^3 = 2^3 = 8$.
  Since $w_0 \in \mathbb{R}\backslash\mathbb{Q}$, we have $g(w_0) = -{w_0}^9 = -(-\sqrt[3]{2})^9 = 8$.
  Then $g(x_0) = g(w_0)$.
  It follows that g is not injective.

ii. We verify that g is surjective:

- Pick any $y \in \mathbb{R}$. Then $\sqrt[3]{y}$, $\sqrt[9]{y}$ are well-defined as real numbers.
  Note that $\sqrt[3]{y}$ is rational or $\sqrt[3]{y}$ is irrational.
  * (Case 1.) Suppose $\sqrt[3]{y}$ is rational.
    Take $t = \sqrt[3]{y}$. By definition, $t \in \mathbb{Q}$. Then $g(t) = t^3 = (\sqrt[3]{y})^3 = y$.
  * (Case 2.) Suppose $\sqrt[3]{y}$ is irrational.
    Take $t = -\sqrt[9]{y}$. By definition, $\sqrt[3]{y} = (-t)^3$.
    Since $\sqrt[3]{y}$ is irrational, $-t$ is also irrational. Then t is irrational.
    We have $g(t) = -t^9 = -(-\sqrt[9]{y})^9 = -(-y) = y$.
    It follows that g is surjective.

16. **Solution.**

(a) i. Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
Suppose $g \circ f$ is surjective.
Pick any $z \in C$.
Since $g \circ f$ is surjective, there exists some $x \in A$ such that $z = (g \circ f)(x)$.
Define $y = f(x)$. We have $y \in B$.
For the same $x \in A$, $y \in B$ and $z \in C$, we have $g(y) = g(f(x)) = (g \circ f)(x) = z$.
It follows that g is surjective.

ii. Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
Suppose $g \circ f$ is injective.
Pick any $x, w \in A$. Suppose $f(x) = f(w)$.
Then $g(f(x)) = g(f(w))$.
Note that $(g \circ f)(x) = g(f(x))$ and $(g \circ f)(w) = g(f(w))$. Then $(g \circ f)(x) = (g \circ f)(w)$.
Since $g \circ f$ is injective, we have $x = w$.
It follows that f is injective.

(b) Let I, J, K be sets, and $\alpha : I \longrightarrow J$, $\beta : J \longrightarrow K$, $\gamma : K \longrightarrow I$ be functions. Suppose $\gamma \circ \beta \circ \alpha$, $\alpha \circ \gamma \circ \beta$ are both injective. Further suppose $\beta \circ \alpha \circ \gamma$ is surjective.
Then:

($\sharp$) Both $\beta \circ \alpha$ and α are injective. (Reason: $\gamma \circ \beta \circ \alpha$ is injective.)

($\natural$) Both $\gamma \circ \beta$ and β are injective. (Reason: $\alpha \circ \gamma \circ \beta$ is injective.)

($\flat$) Both $\beta \circ \alpha$ and β are surjective. (Reason: $\beta \circ \alpha \circ \gamma$ is surjective.)

Therefore, by ($\natural$), ($\flat$) combined, β is both surjective and injective.

Also, by ($\sharp$), ($\flat$) combined, $\beta \circ \alpha$ is both surjective and injective.

By ($\sharp$) alone, α is injective. We verify that α is surjective:

- Pick any $y \in J$. Take $z = \beta(y)$. By definition, $z \in K$.
  Since $\beta \circ \alpha$ is surjective, there exists soem $x \in I$ such that $z = (\beta \circ \alpha)(x)$.
  For the same x, y, z, we have $\beta(y) = z = (\beta \circ \alpha)(x) = \beta(\alpha(x))$.
  Since β is injective, we have $y = \alpha(x)$.
  It follows that α is surjective.

Hence α is both surjective and injective.

We verify that γ is both surjective and injective:

- Pick any $u \in I$. Take $v = \alpha(u)$, $w = \beta(v)$. By definition, $w \in K$.
  Since $\beta \circ \alpha \circ \gamma$ is surjective, there exists some $t \in K$ such that $w = (\beta \circ \alpha \circ \gamma)(t)$.
  For the same $\beta(\alpha(u)) = w = (\beta \circ \alpha \circ \gamma)(t) = \beta(\alpha(\gamma(t)))$
  Since β is injective, we have $\alpha(u) = \alpha(\gamma(t))$. Now, since α is injective, we have $u = \gamma(t)$.
- Pick any $r, s \in K$. Suppose $\gamma(r) = \gamma(s)$.
  Since β is surjective, there exist some $p, q \in J$ such that $r = \beta(p)$ and $s = \beta(q)$.
  Since α is surjective, there exist some $m, n \in I$ such that $p = \alpha(m)$ and $q = \alpha(n)$.
  Then, for the same m, n, p, q, r, s, we have

$$(\gamma \circ \beta \circ \alpha)(m) = \gamma(\beta(\alpha(m))) = \gamma(\beta(p)) = \gamma(r) = \gamma(s) = \gamma(\beta(q)) = \gamma(\beta(\alpha(n))) = (\gamma \circ \beta \circ \alpha)(n).$$

  Since $\gamma \circ \beta \circ \alpha$ is injective, we have $m = n$. Then $r = \beta(p) = \beta(\alpha(m)) = \beta(\alpha(n)) = \beta(q) = s$.
  It follows that γ is injective.

Hence γ is both surjective and injective.

17. **Solution.**

 (a) Take $A = \{0\}$, $B = \{1, 2\}$, $C = \{3\}$. Here $0, 1, 2, 3$ are pairwise distinct objects. Define the function $f : A \longrightarrow B$ by $f(0) = 1$. Define the function $g : B \longrightarrow C$ by $g(1) = g(2) = 3$.

 The function $g \circ f : A \longrightarrow C$ is given by $(g \circ f)(0) = g(f(0)) = g(1) = 3$.

 Pick any $z \in C$. Since $C = \{3\}$, we have $z = 3$. Note that $0 \in A$ and $(g \circ f)(0) = 3 = z$. It follows that $g \circ f$ is surjective.

 Note that $2 \in B$. Pick any $x \in A$. Since $A = \{0\}$, we have $x = 0$. Then $f(x) = f(0) = 1 \neq 2$. Therefore the function f is not surjective.

 (b) Take $A = \{0\}$, $B = \{1, 2\}$, $C = \{3\}$. Here $0, 1, 2, 3$ are pairwise distinct objects. Define the function $f : A \longrightarrow B$ by $f(0) = 1$. Define the function $g : B \longrightarrow C$ by $g(1) = g(2) = 3$.

 The function $g \circ f : A \longrightarrow C$ is given by $(g \circ f)(0) = g(f(0)) = g(1) = 3$.

 Pick any $x, w \in A$. Suppose $(g \circ f)(x) = (g \circ f)(w)$. Since $A = \{0\}$, we have $x = 0 = w$. It follows that $g \circ f$ is injective.

 Note that $1, 2 \in B$, $1 \neq 2$ and $g(1) = g(2) = 3$. Then the function g is not injective.

18. **Solution.**

 (a) Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are surjective.

 Pick any $y \in C \cup D$. By definition of union, $y \in C$ or $y \in D$.

 - (Case 1.) Suppose $y \in C$. By the surjectivity of f, there exists some $x \in A$ such that $y = f(x)$.
 Since $x \in A$, we have $x \in A$ or $x \in B$. Then $x \in A \cup B$. We have $y = f(x) = (f \cup g)(x)$.
 - (Case 2.) Suppose $y \in D$. By the surjectivity of g, there exists some $w \in B$ such that $y = g(w)$.
 Since $w \in B$, we have $w \in A$ or $w \in B$. Then $w \in A \cup B$. We have $y = g(w) = (f \cup g)(w)$.

 It follows that $f \cup g : A \cup B \longrightarrow C \cup D$ is surjective.

 (b) Take $A = \{0\}$, $B = \{1\}$, $C = D = \{2\}$. (Here $0, 1$ are distinct objects.) Define the functions $f : A \longrightarrow C$, $g : B \longrightarrow D$ by $f(0) = 2$, $g(0) = 2$. The functions f, g are injective.

 Note that $A \cap B = \emptyset$. Hence it is (trivially) true that $f(x) = g(x)$ for any $x \in A \cap B$.

 The function $f \cup g : A \cup B \longrightarrow C \cup D$ is given by $(f \cup g)(0) = f(0) = 2$, $(f \cup g)(1) = g(1) = 2$. Note that $0 \neq 1$ and $(f \cup g)(0) = (f \cup g)(1)$. Hence $f \cup g$ is not injective.

19. **Answer.**

 (a) (I) L is a subset of $H \times K$

 (II) a relation

 (III) For any $x \in D$

 (IV) $y \in R$

 (V) $(x, y) \in G$

 (VI) For any $x \in D$, for any $y, z \in R$

 (VII) $(x, y) \in G$ and $(x, z) \in G$

 (VIII) $y = z$

 (IX) D

 (X) R

 (XI) G

 (b) i. $(p, q) = (0, 0)$ or $(p, q) = (0, 1)$.
 $(s, t) = (1, \tau)$, provided that $-2 \leq \tau < 2$.

 ii. $(p, q) = (1, 1)$ and $(s, t) = (2, 1)$.
 Alternative answer: $(p, q) = (1, 2)$ and $(s, t) = (2, 2)$.

 iii. $(m, n) = (0, 0)$.
 $(p, q) = (1, 1)$ or $(p, q) = (1, 2)$.

20. **Answer.**

 (I) F is a subset of $A \times B$

 (II) there exists some $y \in B$

 (III) Pick any $x \in A$.

 (IV) 0

 (V) 4

 (VI) Define $y = 4 + \sqrt{\dfrac{16 - (x - 2)^4}{4}}$

(VII) $y \in B$

(VIII) $(x-2)^4 + 4 \cdot \left[4 + \sqrt{\dfrac{16-(x-2)^4}{4}} - 4\right]^2 = (x-2)^4 + 4 \cdot \dfrac{16-(x-2)^4}{4} = 16$

(IX) $(x, y) \in F$

(X) if $(x, y) \in F$ and $(x, z) \in F$ then $y = z$

(XI) Pick any $x \in A$. Pick any $y, z \in B$. Suppose $(x, y) \in F$ and $(x, z) \in F$.

(XII) $(x-2)^4 + 4(y-4)^2 = 16$

(XIII) since $(x, z) \in F$, we have $(x-2)^4 + 4(z-4)^2 = 16$

(XIV) $\dfrac{16-(x-2)^4}{4}$

(XV) $\sqrt{(y-4)^2} = \sqrt{(z-4)^2} = z - 4$

(XVI) $y = z$

21. (a) **Answer.**

(I) F is a subset of $\mathbb{C} \times E$

(II) relation

(III) for any $z \in \mathbb{C}$, there exists some $w \in E$ such that $(z, w) \in F$

(IV) Pick any $z \in \mathbb{C}$.

(V) well-defined

(VI) Define $w = i + \dfrac{z-1}{1+|z-1|}$

(VII) $\left| i + \dfrac{z-1}{1+|z-1|} - i \right| = \left| \dfrac{z-1}{1+|z-1|} \right| = \dfrac{|z-1|}{1+|z-1|}$

(VIII) $w \in E$

(IX) $(1+|z-1|)(w-i) + 1 = (1+|z-1|) \cdot \left[i + \dfrac{z-1}{1+|z-1|} - i \right] + 1 = z$

(X) for any $z \in \mathbb{C}$, for any $w, v \in E$, if $(z, w) \in F$ and $(z, v) \in F$ then $w = v$

(XI) Pick any

(XII) $\mathbb{C}$

(XIII) E

(XIV) Suppose $(z, w) \in F$ and $(z, v) \in F$

(XV) $[(1+|z-1|)(w-i)+1] - [(1+|z-1|)(v-i)+1] = z - z = 0$

(XVI) $w - v = 0$

(b) **Answer.**

The explicit 'formula of definition' of $f : \mathbb{C} \longrightarrow E$ is given by $f(z) = i + \dfrac{z-1}{1+|z-1|}$ for any $z \in \mathbb{C}$.

(c) **Solution.**

We verify that for any $z, u \in \mathbb{C}$, if $f(z) = f(u)$ then $z = u$:

- Pick any $z, u \in \mathbb{C}$. Suppose $f(z) = f(u)$.
  We have $i + \dfrac{z-1}{1+|z-1|} = f(z) = f(u) = i + \dfrac{u-1}{1+|u-1|}$.
  Then $\dfrac{z-1}{1+|z-1|} = \dfrac{u-1}{1+|u-1|}$. —— $(\star)$
  Therefore $\dfrac{|z-1|}{1+|z-1|} = \left| \dfrac{z-1}{1+|z-1|} \right| = \left| \dfrac{u-1}{1+|u-1|} \right| = \dfrac{|u-1|}{1+|u-1|}$.
  Now $|z-1| + |z-1| \cdot |u-1| = (1+|u-1|) \cdot |z-1| = (1+|z-1|) \cdot |u-1| = |u-1| + |z-1| \cdot |u-1|$.
  Then $|z-1| = |u-1|$. —— $(\star\star)$
  By $(\star), (\star\star)$, we have $\dfrac{z-1}{1+|z-1|} = \dfrac{u-1}{1+|u-1|} = \dfrac{u-1}{1+|z-1|}$. Then $z - 1 = u - 1$. Therefore $z = u$.

(d) **Solution.**

We verify that for any $w \in E$, there exists some $z \in \mathbb{C}$ such that $w = f(z)$:

- Pick any $w \in E$.
  Note that $|w-i|<1$. Then $1-|w-i| \neq 0$. Therefore $\dfrac{w-i}{1-|w-i|}$ is well-defined as a complex number.
  Define $z = 1 + \dfrac{w-i}{1-|w-i|}$. By definition, $z \in \mathbb{C}$.
  We have $|z-1| = \left|\dfrac{w-i}{1-|w-i|}\right| = \dfrac{|w-i|}{1-|w-i|}$.
  We have $f(z) = i + \dfrac{z-1}{1+|z-1|} = i + \dfrac{[1+(w-i)/(1-|w-i|)]-1}{1+|w-i|/(1-|w-i|)} = i + \dfrac{w-i}{(1-|w-i|)+|w-i|} = w$.

(e) **Answer.**

The explicit 'formula of definition' of $f^{-1}: E \longrightarrow \mathbb{C}$ is given by $f^{-1}(w) = 1 + \dfrac{w-i}{1-|w-i|}$ for any $w \in E$.

22. **Solution.**
Let $C = \{(x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } 9x^2+16y^2 = 144\}$.

(a) Let $A = [0,4]$, $B = [0,3]$, and $F = C \cap (A \times B)$.
Define $f = (A,B,F)$. By definition, F is a subset of $A \times B$. Then f is a relation from A to B with graph F.
We verify the statement '*for any $x \in A$, there exists some $y \in B$ such that $(x,y) \in F$*:

- Pick any $x \in A$. We have $0 \leq x \leq 4$. For this x, define $y = \dfrac{3}{4}\sqrt{16-x^2}$. We have $0 \leq y \leq 3$. Then $y \in B$.
  We have $9x^2+16y^2 = 9x^2+16\left(\dfrac{3}{4}\sqrt{16-x^2}\right)^2 = 144$. Hence $\left(x, \dfrac{3}{4}\sqrt{16-x^2}\right) \in F$.

We verify the statement '*for any $x \in A$, for any $y,z \in B$, if $(x,y) \in F$ and $(x,z) \in F$ then $y = z$*:

- Pick any $x \in A$. Pick any $y,z \in B$. Suppose $(x,y) \in F$ and $(x,z) \in F$.
  Since $(x,y) \in F$, we have $9x^2+16y^2 = 144$. Since $(x,z) \in F$, we have $9x^2+16z^2 = 144$. Then $9x^2+16y^2 = 9x^2+16z^2$. We obtain $y^2 = z^2$.
  Since $y,z \in B$, we have $y \geq 0$ and $z \geq 0$. Then $y = \sqrt{y^2} = \sqrt{z^2} = z$.

It follows that f is a function.

(b) Let $A = [2,3]$, $B = [-1,4]$, and $F = C \cap (A \times B)$.
Define $f = (A,B,F)$.
We dis-prove the statement '*for any $x \in A$, for any $y,z \in B$, if $(x,y) \in F$ and $(x,z) \in F$ then $y = z$*:

- Take $x_0 = \dfrac{\sqrt{8\sqrt{2}}}{3}$, $y_0 = -1$, $z_0 = 1$.
  Note that $x_0{}^2 = \dfrac{128}{9}$. Then $4 \leq x_0{}^2 \leq 9$. Therefore $2 \leq x_0 \leq 3$. Hence $x_0 \in A$.
  Also note that $y_0, z_0 \in B$ and $z_0 \in B$, and $y_0 \neq z_0$.
  We have $9x_0{}^2 + 16y_0{}^2 = 144$. Then $(x_0, y_0) \in F$.
  We have $9x_0{}^2 + 16z_0{}^2 = 144$. Then $(x_0, z_0) \in F$.

Hence f is not a function.

(c) Let $A = [1,4]$, $B = [0,5/2]$, and $F = C \cap (A \times B)$.
Define $f = (A,B,F)$.
We dis-prove the statement '*for any $x \in A$ there exists $y \in B$ such that $(x,y) \in F$*:

- Take $x_0 = 1$. Note that $x_0 \in A$.
  For any $y \in B$, we have $0 \leq y \leq \dfrac{5}{2}$. Then $y^2 \leq \dfrac{25}{4}$.
  Therefore $9 \cdot x_0{}^2 + 16y^2 = 9 + 16y^2 \leq 9 + 16 \cdot \dfrac{25}{4} = 109 < 144$. Then $(x_0, y) \notin F$.

Hence f is not a function.

23. **Solution.**
Let $I = (0, +\infty)$.

Let $F = \left\{ (p,z) \,\middle|\, \begin{array}{l} z \in I \text{ and} \\ \text{there exist some } x,y \in I \text{ such that } p = (x,y) \text{ and } z^2+2(x+y)z = x^2+y^2 \end{array} \right\}$.

Define $f = (I^2, I, F)$.

(a) By definition, $F \subset I^2 \times I$. Then f is a relation from I^2 to I with graph F.

We verify that for any $P \in I^2$, there exists some $z \in I$ such that $(p, z) \in F$:

- Pick any $p \in I^2$. There exist some $x, y \in I$ such that $p = (x, y)$.
  Note that $x^2 + y^2 + (x + y)^2 > (x + y)^2 > 0$.
  Then $\sqrt{x^2 + y^2 + (x + y)^2}$ is well-defined as a real number. Also, $\sqrt{x^2 + y^2 + (x + y)^2} > x + y$. —— $(\star)$
  Define $z = -(x + y) + \sqrt{x^2 + y^2 + (x - y)^2}$. By $(\star)$, z is well-defined as a positive real number. Then $z \in I$.
  We have $z^2 + 2(x + y)z + (x + y)^2 = [z + (x + y)]^2 = \left(\sqrt{x^2 + y^2 + (x + y)^2}\right)^2 = x^2 + y^2 + (x + y)^2$.
  Then $z^2 + 2(x + y)z = x^2 + y^2$. Therefore $(p, z) \in F$.

We verify that for any $p \in I^2$, for any $z, w \in I$, if $(p, z) \in F$ and $(p, w) \in F$ then $z = w$:

- Pick any $p \in I^2$. Pick any $z, w \in I$. Suppose $(p, z) \in F$ and $(p, w) \in F$.
  Since $(p, z) \in F$, there exist some $x, y \in I$ such that $p = (x, y)$ and $z^2 + 2(x + y)z = x^2 + y^2$.
  Since $(p, w) \in F$, there exist some $u, v \in I$ such that $p = (u, v)$ and $w^2 + 2(u + v)w = u^2 + v^2$.
  Since $p = (x, y)$ and $p = (u, v)$, we have $(x, y) = (u, v)$. Then $x = u$ and $y = v$.
  Then $z^2 + 2(x + y)z = x^2 + y^2 = u^2 + v^2 = w^2 + 2(u + v)w = w^2 + 2(x + y)w$.
  Therefore $(z - w)[z + w + 2(x + y)] = z^2 - w^2 + 2(x + y)z - 2(x + y)w = 0$.
  Since $z, w, x, y \in I$, $z + 2 + 2(x + y)$ is a positive real number. Then $z - w = 0$. Therefore $z = w$.

It follows that f is a function from I^2 to I with graph F.

(b) i. The explicit 'formula of definition' of $f : I^2 \longrightarrow I$ is given by $f(x, y) = -(x + y) + \sqrt{x^2 + y^2 + (x + y)^2}$ for any $x, y \in \mathbb{R}$.

ii. Note that for any $t \in I$, we have $f(t, t) = -(t + t) + \sqrt{t^2 + t^2 + (t + t)^2} = -2t + \sqrt{t^2 + t^2 + 4t^2} = (\sqrt{6} - 2)t$.
We verify that for any $z \in I$, there exists some $p \in I^2$ such that $z = f(p)$:

- Pick any $z \in I$.
  Take $x = \dfrac{z}{\sqrt{6} - 2}$. By definition $x \in I$. Then $(x, x) \in I^2$.
  We have $f(x, x) = (\sqrt{6} - 2)x = (\sqrt{6} - 2) \cdot \dfrac{z}{\sqrt{6} - 2}$.

It follows that f is surjective.

iii. We verify that there exist some $p_1, p_2 \in I^2$ such that $p_1 \neq p_2$ and $f(p_1) = f(p_2)$:

- Take $x_1 = 1, y_1 = 2, x_2 = 2, y_2 = 1$.
  We have $(x_1, y_1) \in I^2$ and $(x_2, y_2) \in I^2$. Also, $(x_1, y_1) \neq (x_2, y_2)$.
  Note that $x_1 = y_2$ and $x_2 = y_1$.
  We have $f(x_1, y_1) = -(x_1 + y_1) + \sqrt{{x_1}^2 + {y_1}^2 + (x_1 + y_1)^2} = -(y_2 + x_2) + \sqrt{{y_2}^2 + {x_2}^2 + (y_2 + x_2)^2} = -(x_2 + y_2) + \sqrt{{x_2}^2 + {y_2}^2 + (x_2 + y_2)^2} = f(x_2, y_2)$.

It follows that f is not injective.