

Advice.

1. The questions in this assignment are mostly about surjectivity, injectivity, inverse function, the formal definition of functions as relations. Do familiarize yourself with the corresponding material available in the course homepage before trying the questions.
2. Questions which require more thought and/or work and/or tricks and/or organization and/or ... than the ‘unlabelled’ questions are labelled by $\diamond$, $\clubsuit$, $\heartsuit$, $\spadesuit$ in ascending order of overall difficulty level.

Instructions.

1. Any work submitted by you must be written on **A4-size sheets** and must be appropriately binded.
Your name and student ID, as in your student card, and the **code** of the section to which you are registered must be written at the upper right corner of the first page of your submission.
2. (a) **Mandatory work, for assessment purpose.**
 You are **required** to submit work on Questions (1), (3), (5), (6), (7a), (7b), (8), (9), (10), (12a), (19), (20), (21) for course assessment purpose.
 (b) **Optional proof-writing exercise.**
 You may also **opt** to submit, **on exactly one sheet, separate from your submission on mandatory work**, your work on Questions (16a), (17). It will be read and commented, but not counted for course assessment.
 (c) **Other optional work.**
 You may choose to submit work on other questions in this assignment not mentioned above, alongside the mandatory work, but there is no guarantee that it will be read.
3. (a) You must adhere to the notations which have been introduced in the course. Things which have not been formally defined in the course are not allowed in your work.
 (b) Words of the likes of ‘*trivial*’, ‘*obvious*’, ‘*clear(-ly)*’ are not allowed to appear in your work.

* * *

1. Denote by (J) the statement below:—
 (J) Let ζ be a complex number. Suppose ζ^2 is not a real number. Then for any complex number z , there are at most one real number a and at most one real number b satisfying $z = a\zeta + b\bar{\zeta}$.
 (a) Fill in the blanks in the passage below so as to obtain a re-formulation, labelled (J') , of the statement (J) :—
 (J') Let ζ be a complex number. Suppose ζ^2 is not a real number. Then for any complex number z , (I) ,
 if $z = a\zeta + b\bar{\zeta}$ and (II) then (III) = a' and (IV) = b' .
 (b) $\diamond$ Prove the statement (J) (with reference to its re-formulation (J')).
2. $\diamond$ Prove the statement $(\natural)$, with direct reference to the definition of *strict monotonicity*:—
 $(\natural)$ Let I be an interval in $\mathbb{R}$, and $f, g : I \rightarrow \mathbb{R}$ be functions. Suppose f is strictly increasing on I and g is strictly decreasing on I . Then there is at most one $c \in I$ satisfying $f(c) = g(c)$.

Remarks.

- Be careful. The functions under consideration in the statement $(\natural)$ are not assumed to be differentiable. Whatever you have learnt in your *calculus* course which relates (strict) monotonicity with derivatives is unapplicable here.
 - At some stage of the argument, you may find it more convenient to apply the method of proof-by-contradiction.
 - You are advised to re-formulate the conclusion part of the statement $(\natural)$ appropriately before trying to deduce it from the assumption in the statement $(\natural)$.
3. (a) Denote the statement below by (M) :—
 (M) Suppose A is a set, and $f, g : A \rightarrow A$ are functions. Then the equality $g \circ f = f \circ g$ as functions holds.
 Write down the negation $(\sim M)$ of the statement (M) .
 (b) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by $f(x) = \frac{x^2}{1+x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

- i. Write down the respective ‘formulae of definition’ of the functions $g \circ f$, $f \circ g$ explicitly.
- ii. Name an appropriate real number x_0 for which $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Justify your answer.
- iii. Is it true that $g \circ f = f \circ g$ as functions? Justify your answer.

Remark. Hence we have dis-proved the statement (M) by providing a counter-example against it. (Why?)

- (c) Define $A = \{0, 1\}$. Name a pair of functions $f, g : A \rightarrow A$ for which $g \circ f \neq f \circ g$ as functions. Justify your answer.

Remark. Hence we have dis-proved the statement (M) with another counter-example against it. (Why?)

4. ♣ In this question, take for granted what you have learnt in the calculus of one real variable.

We introduce/recall the definition for the notion of *smoothness* for real-valued functions of one real variable:—

A real-valued function of one real variable is said to be **smooth** if it is differentiable for as many times as we like at every point of its domain.

Denote by $C^\infty(\mathbb{R})$ the set of all smooth real-valued functions on $\mathbb{R}$. Define the functions $X, D, I_0 : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ by

$$(X(\varphi))(x) = x\varphi(x), \quad (D(\varphi))(x) = \varphi'(x), \quad (I_0(\varphi))(x) = \int_0^x \varphi(t)dt \quad \text{for any } \varphi \in C^\infty(\mathbb{R}) \text{ for any } x \in \mathbb{R}.$$

- (a) Suppose $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\alpha(x) = \frac{2x}{x^2 + 1}$ for any $x \in \mathbb{R}$. What are $X(\alpha)$, $D(\alpha)$, $I_0(\alpha)$ respectively?

- (b) Verify the statements below:—

- i. For any $\varphi \in C^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, $((D \circ X)(\varphi))(x) - ((X \circ D)(\varphi))(x) = \varphi(x)$.
- ii. For any $\varphi \in C^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, $((X \circ I_0)(\varphi))(x) - ((I_0 \circ X)(\varphi))(x) = ((I_0 \circ I_0)(\varphi))(x)$.

Remark. The seemingly innocuous mathematical statement given in part (b.i) is a baby case of something of great significance in *modern physics*; it is behind the **Heisenberg relations** for position and momentum in *quantum mechanics*.

5. (a) Explain the phrases below by stating the appropriate definitions.

- i. *surjective function*
- ii. *injective function*

- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^{\frac{5}{3}} - 1$ for any $x \in \mathbb{R}$.

Denote by (A), (B) the statements below:— (A) f is surjective. (B) f is injective.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs for the statements (A), (B) in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

- i. Here we prove the statement (A):—

[We want to verify that f is surjective.
This amounts to verifying the statement ‘ (I) $y \in \mathbb{R}$, (II) $x \in \mathbb{R}$ such that (III) .’]
(IV) $y \in \mathbb{R}$. Take (V) . By definition $x \in \mathbb{R}$. Also note that (VI) .
It follows that (VII) .

- ii. Here we prove the statement (B):—

[We want to verify that f is injective.
This amounts to verifying the statement ‘ (I) $x, w \in \mathbb{R}$, (II) then $x = w$.’]
Pick any (III) . (IV) $f(x) = f(w)$.
Then $x^{\frac{5}{3}} =$ (V) $= w^{\frac{5}{3}}$. Since $x, w \in \mathbb{R}$, we have $x = (x^{\frac{5}{3}})^{\frac{3}{5}} =$ (VI) .
It follows that (VII) .

- (c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{x}{x^2 + 1}$ for any $x \in \mathbb{R}$.

Denote by (C), (D) the statements below:— (C) f is surjective. (D) f is injective.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* dis-proofs against the statements (C), (D) in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* dis-proof against each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

i. Here we dis-prove the statement (C):—

[We want to verify that f is not surjective. This amounts to verifying the statement ‘ (I) $y_0 \in \mathbb{R}$ such that (II) $x \in \mathbb{R}$ such that (III) .’] (IV) $y_0 = 1$. We verify, using the method of proof-by-contradiction, that (V) , $f(x) \neq y_0$:— * (VI) it were true that (VII) . Then $\frac{x_0}{x_0^2 + 1} = f(x_0) = y_0 = 1$. Therefore (VIII) $= x_0^2 - x_0 + 1 =$ (IX) > 0. Contradiction arises. It follows that (X) .
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ii. Here we dis-prove the statement (D):—

[We want to verify that f is not injective. This amounts to verifying the statement ‘ (I) $x_0, w_0 \in \mathbb{R}$ such that $f(x_0)$ (II) and (III) .’] Take $x_0 = \frac{1}{2}$, (IV) . Note that $x_0, w_0 \in \mathbb{R}$. Also note that (V) . We have $f(x_0) = \frac{1/2}{(1/2)^2 + 1} = \frac{2}{5}$ and (VI) . Then $f(x_0) =$ (VII) . It follows that (VIII) .

6. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be the function defined by $f(z) = z^5$ for any $z \in \mathbb{C}$.

Denote by (E), (F) the statements below:— (E) f is surjective. (F) f is not injective.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs of the statements (E), (F) in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

(a) Here we prove the statement (E):—

[We want to verify the statement ‘ (I) .’] (II) . For this ζ, (III) such that $\zeta = \zeta \cdot (\cos(\theta) + i \sin(\theta))$. (IV) By definition, $z \in \mathbb{C}$. Note that $f(z) = z^5 =$ (V) . It follows that (VI) .

(b) Here we prove the statement (F):—

[We want to verify the statement ‘ (I) .’] (II) . Note that $z_0, w_0 \in \mathbb{C}$. Also note that (III) . [We evaluate $f(z_0), f(w_0)$.] We have (IV) and (V) . Then $f(z_0) = f(w_0)$. It follows that (VI) .
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7. (a) Explain the phrases below by stating the appropriate definitions.

i. *composition of functions*

ii. *identity function on a set*

(b) $\diamond$ Denote by (T) the statement below:—

(T) Let A be a set, and $f : A \rightarrow A$ be a function. Suppose $f \circ f = f$. Further suppose (f is injective or f is surjective). Then $f = \text{id}_A$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs of the statement (T) in the block below, with appropriate words/symbols so as to obtain a *complete* proof for the statement (T).

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

Let A be a set, and $f : A \rightarrow A$ be a function.

Suppose $f \circ f = f$. — (★)

(I)

[We want to verify that $f = \text{id}_A$. This amounts to verifying ‘for any $x \in A$, $f(x) = \text{id}_A(x)$ ’.]

- (Case 1.) Suppose f is injective.

Pick any (II). By the definition of the function f , we have $f(x) \in A$.

By (★), we have $(f \circ f)(x) =$ (III).

By the definition of composition, we have (IV) $= f(f(x))$.

Then $f(f(x)) = f(x)$. Now, by (V), we have (VI).

It follows that $f = \text{id}_A$.

- (Case 2.) (VII)

(VIII) $x \in A$. By the definition of surjectivity, (IX).

Then we have $f(x) = f(f(u)) =$ (X) by the definition of composition.

By (★), we have (XI) $= x$. Then $f(x) = x = \text{id}_A(x)$.

It follows that (XII).

Hence, in any case, $f = \text{id}_A$.

(c)♣ Hence, or otherwise, prove the statement (♯):—

(♯) Let B be a set, K be a subset of B , and $\varphi : \mathfrak{P}(B) \rightarrow \mathfrak{P}(B)$ be the function defined by $\varphi(S) = S \cap K$ for any $S \in \mathfrak{P}(B)$. Suppose φ is injective or φ is surjective. Then $K = B$.

8.♣ We introduce the notation for the set of all functions from a given set to a given set:—

Let D, R be sets. We denote by $\text{Map}(D, R)$ the set of all functions with domain D and range R .

Let A, B be non-empty sets. For any $x \in A$, define the function $E_x : \text{Map}(A, B) \rightarrow B$ by $E_x(f) = f(x)$ for any $f \in \text{Map}(A, B)$.

Denote by (P), (Q) the statements below:—

(P) For any $x \in A$, the function E_x is surjective.

(Q) Suppose B has more than one element. Also suppose there exists some $u \in A$ such that E_u is injective. Then A is a singleton.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs of the statements (P), (Q) in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

(a) Here we prove the statement (P):—

(I). We verify that E_x is surjective:

- (II). Define the function $f : A \rightarrow B$ by (III).

By definition, $f \in$ (IV). By definition of E_x , we have (V).

It follows that E_x is surjective.

(b) Here we prove the statement (Q):—

Suppose B has more than one element. Pick (I) $y, z \in B$.

Also suppose (II) $\quad$.

Note that $\{u\} \subset A$. We now verify $A \subset \{u\}$:

- Pick any $x \in A$. Suppose it were true that $x \notin \{u\}$. Then by definition of complement, (III) $\quad$.

(IV) $f(t) = y$ (V) $t \in A$.

Define (VI) $\quad$ by $g(t) = \begin{cases} y & \text{if } \text{(VII)} \\ z & \text{if } t \in A \setminus \{u\} \end{cases}$.

By definition, $f, g \in \text{Map}(A, B)$.

We have $E_u(f) = \text{(VIII)} \quad$. Then, since E_u is injective, we have (IX) $\quad$.

Recall that $x \in A \setminus \{u\}$. Since $f(x) = y$ and $g(x) = z$ and (X) $\quad$, we have (XI) $\quad$.

Now $f = g$ (XII) $f \neq g$. Contradiction arises.

It follows that in the first place, we have $x \in \{u\}$.

Hence $A = \{u\}$.

9. (a) Explain the phrase ‘an inverse function of a function’ by providing an appropriate definition.

(b) *You are not required to prove your answers in this part of the question.*

The function $f : [2, 3] \rightarrow [a, b]$ given by $f(x) = x^4 - 8x^2 + 7$ for any $x \in [2, 3]$ is known to be a bijective function from the interval $[2, 3]$ to the interval $[a, b]$. Here a, b are some appropriate real numbers.

(You may take for granted that f is a strictly increasing function on $[2, 3]$.)

- What are the respective values of a, b ?
- Write down the explicit ‘formula of definition’ for the inverse function f^{-1} of the function f .
- Consider each expression below.
 - * If the expression is well-defined as a number, write ‘well-defined’ and give its explicit value.
 - * If the expression is not well-defined as a number, just write ‘not well-defined’.

A. $f^{-1}(f(\sqrt{5}))$.

B. $f(f^{-1}(f(f^{-1}(\pi))))$.

10. *You are not required to prove your answers in this question.*

The function $f : (0, +\infty) \rightarrow J$, given by $f(x) = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{e^{\sqrt{x}} - e^{-\sqrt{x}}}$ for any $x \in (0, +\infty)$ is known to be a bijective function from $(0, +\infty)$ to the set J .

- Express the set J explicitly as an interval.
- Write down the explicit ‘formula of definition’ for the inverse function f^{-1} of the function f .

11. *Do not use any result from calculus of one real variable.*

Denote the interval $(0, +\infty)$ by I . Let $f : I \rightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{1}{2} \left(x - \frac{1}{x} \right)$ for any $x \in I$.

- Verify that f is injective, with reference to the definition of injectivity.
- Verify that f is surjective, with reference to the definition of surjectivity.
- Write down the explicit ‘formula of definition’ for the inverse function f^{-1} of the function f .

12. *In this question, you will want to make good use of the basic properties of conjugate and modulus for complex numbers. You are advised against using arguments of complex numbers, and also against ‘decomposing’ complex numbers into real and imaginary parts. In fact, there is no need to use such things.*

- Let $h : \mathbb{C} \rightarrow \mathbb{C}$ be the function defined by $h(z) = \bar{z}$ for any $z \in \mathbb{C}$.
 - Verify that h is bijective with reference to the respective definitions of surjectivity and injectivity.
 - Write down the ‘formula of definition’ of the inverse function of h .
- ♣ Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be the function defined by $f(z) = z^2 \bar{z}$ for any $z \in \mathbb{C}$.

- i. Verify that f is surjective, with reference to the definition of surjectivity.
 - ii. Verify that f is injective, with reference to the definition of injectivity.
 - (c)[◇] Write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Let $g : \mathbb{C}^* \rightarrow \mathbb{C}$ be the function defined by $g(z) = z/\bar{z}$ for any $z \in \mathbb{C}^*$.
 - i. Verify that g is not injective, with reference to the definition of injectivity.
 - ii. Verify that g is not surjective, with reference to the definition of surjectivity.
13. Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

- (a) Prove that for any $z \in \mathbb{C}$, $\frac{az+b}{cz+d} \neq \frac{a}{c}$.
- (b) Define the function $f : \mathbb{C} \setminus \{-d/c\} \rightarrow \mathbb{C} \setminus \{a/c\}$ by $f(z) = \frac{az+b}{cz+d}$ for any $z \in \mathbb{C} \setminus \{-d/c\}$.
 - i. Verify that f is injective, with reference to the definition of injectivity.
 - ii.[◇] Verify that f is surjective, with reference to the definition of surjectivity.
 - iii. Write down the ‘formula of definition’ of the inverse function of f .

14. Do not use any result from calculus of one real variable.

Let $f : (0, +\infty) \rightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{x^2 - 2x + 4}{x^2 + 2x + 4} \cos\left(\frac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

- (a) Verify that f is not injective, with reference to the definition of injectivity.
 - (b)
 - i. Verify that $\left| \frac{x^2 - 2x + 4}{x^2 + 2x + 4} \right| \leq 1$ for any $x \in (0, +\infty)$.
 - ii. Apply the previous part, or otherwise, to verify that f is not surjective, with reference to the definition of surjectivity.
15. (a)[♣] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} x^9 & \text{if } x \in \mathbb{Q} \\ -x^3 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.
- i. Is f injective? Justify your answer with reference to the definition of injectivity.
 - ii. Is f surjective? Justify your answer with reference to the definition of surjectivity.
- (b)[♣] Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $g(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{Q} \\ -x^9 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.
- i. Is g injective? Justify your answer with reference to the definition of injectivity.
 - ii. Is g surjective? Justify your answer with reference to the definition of surjectivity.

16. (a) Prove each of the statements below:—
- i. Let A, B, C be sets, and $f : A \rightarrow B, g : B \rightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then g is surjective.
 - ii. Let A, B, C be sets, and $f : A \rightarrow B, g : B \rightarrow C$ be functions. Suppose $g \circ f$ is injective. Then f is injective.
- (b)[♣] Let I, J, K be sets, and $\alpha : I \rightarrow J, \beta : J \rightarrow K, \gamma : K \rightarrow I$ be functions. Suppose $\gamma \circ \beta \circ \alpha, \alpha \circ \gamma \circ \beta$ are both injective. Further suppose $\beta \circ \alpha \circ \gamma$ is surjective.
- Prove that each of the functions α, β, γ is both surjective and injective.

17. Dis-prove each of the statements below by giving an appropriate argument:—

- (a) Let A, B, C be sets, and $f : A \rightarrow B, g : B \rightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then f is surjective.
- (b) Let A, B, C be sets, and $f : A \rightarrow B, g : B \rightarrow C$ be functions. Suppose $g \circ f$ is injective. Then g is injective.

18.[◇] We introduce this definition for the notion of *union of functions* below:

Let A, B, C, D be sets, and $f : A \rightarrow C, g : B \rightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Define the function $f \cup g : A \cup B \rightarrow C \cup D$ by $(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$.

The function $f \cup g$ is called the **union of the functions** f, g .

Consider each statement below. Determine whether it is true or false. Justify your answer with an appropriate argument.

- (a) Let A, B, C, D be sets, and $f : A \rightarrow C, g : B \rightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are surjective. Then $f \cup g : A \cup B \rightarrow C \cup D$ is surjective.

- (b) Let A, B, C, D be sets, and $f : A \rightarrow C, g : B \rightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are injective. Then $f \cup g : A \cup B \rightarrow C \cup D$ is injective.
19. (a) Fill in the blanks in the passage below so as to give the respective definitions for the notions of *relation* and *function*:
- Suppose H, K, L be sets. Then we say that (H, K, L) is a **relation** if _____ (I) .
 - Suppose D, R, G are sets. Then we say that (D, R, G) is a **function** if (D, R, G) is _____ (II) and the statements (E), (U) below hold:—
 (E): _____ (III) , there exists some _____ (IV) such that _____ (V) .
 (U): _____ (VI) , if _____ (VII) then _____ (VIII) .
 For such a function, we say that _____ (IX) is its domain, _____ (X) is its range, and _____ (XI) is its graph.
- (b) You are not required to justify your answers in this question. In each part, you are only required to give one correct answer, although there are different correct answers.
- Let $A = [-1, 1], B = [-2, 2], G = \{(x, x) \mid x \leq 0\}, H = \{(x, x+1) \mid x \geq 0\}$ and $F = (A \times B) \cap (G \cup H)$. Name some appropriate $(p, q), (s, t) \in A \times B$, if such exist, for which the ordered triple $(A, B, (F \setminus \{(p, q)\}) \cup \{(s, t)\})$ is a function from A to B .
 - Let $A = [0, 2], G = \{(x, x^2) \mid 0 \leq x \leq 1\}, H = \{(x, 3-x) \mid 1 \leq x < 2\}$ and $F = A^2 \cap (G \cup H)$. Name some appropriate $(p, q), (s, t) \in A^2$, if such exist, for which the ordered triple $(A, A, (F \setminus \{(p, q)\}) \cup \{(s, t)\})$ is an injective function from A to A .
 - Let $A = [0, +\infty)$ and E, F be the subsets of $\mathbb{R}^2$ defined respectively by $E = \{(x, x^{-1}) \mid 0 < x \leq 1\}, F = \{(x, 2x^{-2}) \mid x \geq 1\}$. Name some appropriate $(m, n), (p, q) \in A^2$, if such exist, for which the ordered triple $(A, A, (E \cup F \cup \{(m, n)\}) \setminus \{(p, q)\})$ is a surjective function from A to A .
20. $\diamond$ Let $A = [0, 4], B = [4, 6]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } (x-2)^4 + 4(y-4)^2 = 16\}$. Define $f = (A, B, F)$.

Denote by (G) the statement below:—

(G) f is a function from A to B with graph F .

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proof of the statement (G) in the block below, with appropriate words/symbols so as to obtain a *complete* proof for the statement (G) .

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

By definition, _____ (I) . Then f is a relation from A to B with graph F .

We verify the statement ‘for any $x \in A$, _____ (II) such that $(x, y) \in F$ ’:

- _____ (III)

By definition, $0 \leq x \leq 4$. Then $-2 \leq x-2 \leq 2$. Therefore $0 \leq (x-2)^4 \leq 16$.

Hence _____ (IV) $\leq \frac{16 - (x-2)^4}{4} \leq$ _____ (V) .

_____ (VI) . By definition, $4 \leq y \leq 6$. Then _____ (VII) .

Also by definition, $(x-2)^4 + 4(y-4)^2 =$ _____ (VIII) .

Hence _____ (IX) .

We verify the statement ‘for any $x \in A$, for any $y, z \in B$, _____ (X) ’:

- _____ (XI)

Since $(x, y) \in F$, we have _____ (XII) .

Also, _____ (XIII) .

Then $(y-4)^2 =$ _____ (XIV) $= (z-4)^2$.

Since $y, z \in B$, we have $y-4 \geq 0$ and $z-4 \geq 0$.

Then $y-4 =$ _____ (XV) .

Therefore _____ (XVI) .

It follows that f is a function.

21. Let $E = \{w \in \mathbb{C} : |w - i| < 1\}$, and $F = \{(z, w) \mid z \in \mathbb{C} \text{ and } w \in E \text{ and } (1 + |z - 1|)(w - i) + 1 = z\}$. Define $f = (\mathbb{C}, E, F)$.

(a)[◇] Denote by (H) the statement below:—

(H) f is a function from $\mathbb{C}$ to E with graph F .

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proof of the statement (H) in the block below, with appropriate words/symbols so as to obtain a *complete* proof for the statement (H) .

(The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

By definition, _____ (I) . Then f is a _____ (II) from $\mathbb{C}$ to E with graph F .

• We verify that _____ (III) :

_____ (IV)

Note that $|z - 1| \geq 0$. Then $1 + |z - 1| > |z - 1| \geq 0$. Therefore $1 + |z - 1| \neq 0$.

Hence $\frac{z - 1}{1 + |z - 1|}$ is _____ (V) as a complex number.

Moreover, $0 \leq \frac{|z - 1|}{1 + |z - 1|} < 1$. — (*)

_____ (VI) . By definition, $w \in \mathbb{C}$.

We have $|w - i| = \frac{1}{1 + |z - 1|} < 1$. (The last inequality holds by (*).)

Then _____ (VIII) .

We have _____ (IX) .

Therefore $(z, w) \in F$.

• We verify that _____ (X) :

_____ (XI) $z \in$ _____ (XII) , $w, v \in$ _____ (XIII) . _____ (XIV) .

Then $(1 + |z - 1|)(w - i) + 1 = z$ and $(1 + |z - 1|)(v - i) + 1 = z$.

Then $(1 + |z - 1|)(w - v) = \frac{0}{1 + |z - 1|} = 0$. _____ (XV)

We have $|z - 1| \geq 0$. Then $1 + |z - 1| > 0$. Therefore $1 + |z - 1| \neq 0$.

Hence _____ (XVI) . Then $w = v$.

Hence f is a function.

(b) Write down the explicit ‘formula of definition’ for the function f .

(c)[◇] Verify that f is injective.

(d)[♣] Verify that f is surjective.

(e) Write down the explicit ‘formula of definition’ for the inverse function f^{-1} of the function f .

22.[◇] Let $C = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } 9x^2 + 16y^2 = 144\}$.

(a) Let $A = [0, 4]$, $B = [0, 3]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.

Verify that f is a function.

(b) Let $A = [2, 3]$, $B = [-1, 4]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.

Is f a function? Justify your answer.

(c) Let $A = [1, 4]$, $B = [0, 5/2]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.

Is f a function? Justify your answer.

23.[♣] Let $I = (0, +\infty)$. (So by definition, $I^2 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x > 0 \text{ and } y > 0\}$.)

Let $F = \left\{ (p, z) \mid \begin{array}{l} z \in I \text{ and} \\ \text{there exist some } x, y \in I \text{ such that } p = (x, y) \text{ and } z^2 + 2(x + y)z = x^2 + y^2 \end{array} \right\}$.

Define $f = (I^2, I, F)$.

(a) Prove that f is a function from I^2 to I with graph F .

(b) i. Write down the explicit ‘formula of definition’ for the function f .

ii. Is f surjective? Justify your answer with reference to the definition of surjectivity.

iii. Is f injective? Justify your answer with reference to the definition of injectivity.