MATH1050BC/1058 Assignment 4 (Answers and selected solution)

1. **Answer.**
   (a) $(\exists a)(\forall b)(\exists c)[P(c) \wedge [(\sim Q(a)) \vee (\sim R(b))]]$.
   (b) $(\forall a)(\exists b)(\forall c)[(\sim P(a,c)) \vee [Q(a,b,c) \wedge (\sim R(a,b,c))]]$.
   (c) $(\exists a)(\exists b)(\forall c)(\forall d)[[(\sim P(a,c)) \wedge (\sim Q(b,d))] \vee (\sim R(c,d))]$.
   (d) $(\exists a)(\exists b)[P(a,b) \wedge [(\forall c)(\exists d)[(\sim Q(a,b,c,d)) \vee (\sim R(a,b,c,d))]]]$.
   (e) $(\exists a)(\forall b)[[(\exists c)(\forall d)[P(a,b,c,d) \wedge (\sim Q(a,b,c,d))]] \vee [(\exists e)(\forall f)[(\sim S(a,b,e,f)) \wedge (\sim T(a,b,e,f))]]]$.
   (f) $(\exists a)[[(\exists b)(\forall c)P(b,c) \longrightarrow Q(a,b,c)] \wedge [(\exists d)(\forall e)[R(a,d,e) \wedge [(\sim S(a,d,e)) \vee (\sim T(a,d,e))]]]]$.

2. **Answer.**
   (a) *For any $\zeta \in \mathbb{C}$, there exists some $\eta \in \mathbb{C}$ such that ($|\zeta| \geq |\eta|$ or $|\zeta+\eta| > |\zeta-\eta|$).*
   (b) There exists some $x \in \mathbb{R}$ such that for any $s,t \in \mathbb{Q}$, there exists some $n \in \mathbb{Z}$ such that $s < n < t$ and $|x-n| \leq |t-s|$.
   (c) There exists some $p \in \mathbb{R}$, such that for any $q \in \mathbb{R}$, $n \in \mathbb{N}$, there exist some $s,t \in \mathbb{R}$ such that $|s-t| < |q|$ and $|s^n - t^n| \geq |p|$.
   (d) There exists some $s,t \in \mathbb{Q}$ such that for any $p,q \in \mathbb{R}$, there exists some $n \in \mathbb{Z}$ such that $|s-t| \leq |q|$ and ($t^n > |p|$ or $s^n > |p|$).
   (e) For any $n \in \mathbb{N}$, there exists some $\varepsilon \in (0,+\infty)$ such that for any $\delta \in (0,+\infty)$, there exist some $u,v \in \mathbb{C}$ such that $|u-v| < \delta$ and $|u^n - v^n| \geq \varepsilon$.
   (f) There exist some $p,q \in \mathbb{Z}$ such that for any $s,t \in \mathbb{Z}$, there exist some $m,n \in \mathbb{N}$ such that $|p+q| \geq s^m$ and $|p^n - q| \geq t$ and $|p-q^n| \geq t$.
   (g) There exists some $z \in \mathbb{C}$ such that for any $r \in \mathbb{R}$, there exists some $w \in \mathbb{C}$ such that ($|z-w| \leq r$ and ($z \in \mathbb{R}$ or $|w| \leq r$)).
   (h) There exist some $z,w \in \mathbb{C}$ such that $|z-w| \geq |z+w|$ and (for any $s \in \mathbb{R}$, there exists some $t \in \mathbb{R}$ such that ($|z-s-t| \leq w$ and $|z| \geq 1$)).
   (i) There exist some $\zeta,\alpha,\beta \in \mathbb{C}$ such that (there exist some $s,t \in \mathbb{R}$ such that $\zeta = s\alpha + t\beta$) and (for any $p,q \in \mathbb{R}$, $\zeta \neq p\bar{\alpha} + q\bar{\beta}$.)
   (j) There exist some $\zeta,\alpha,\beta,\gamma \in \mathbb{C}$ such that $\alpha\zeta^2 + \beta\zeta + \gamma = 0$ and (there exist some $s,t \in \mathbb{R}$ such that for any $r \in \mathbb{R}$, $\zeta \neq r\alpha + s\beta + t\gamma$).
   (k) There exist some $\zeta \in \mathbb{C}$ such that (for any $\alpha \in \mathbb{C}$, there exist some $\beta,\gamma \in \mathbb{C}$ such that $\alpha\zeta^2 + \beta\zeta + \gamma = 0$) and (there exist some $\rho,\sigma \in \mathbb{C}$ and some $s,t \in \mathbb{R}$ such that for any $\tau \in \mathbb{C}$, for any $r \in \mathbb{R}$, $\zeta \neq r\rho + s\sigma + t\tau$).
   (l) There exists some $\zeta \in \mathbb{C}$ such that $|\zeta| > 1$ and (for any $\beta,\gamma \in \mathbb{C}$, if $\beta\gamma \neq 0$ then there exists some $\alpha \in \mathbb{C}$ such that $\alpha\zeta^2 + \beta\zeta + \gamma = 0$) and (there exist some $\rho,\sigma \in \mathbb{C}$ and some $s,t \in \mathbb{R}$ such that ($st \neq 0$ and $\rho + \sigma \neq 0$) and (for any $\tau \in \mathbb{C}$, for any $r \in \mathbb{R}$, $\zeta \neq r\rho + s\sigma + t\tau$)).

3. **Solution.**
   (a) Take $n = 3$. By definition, $n \in \mathbb{N}$. Note that $n+2 = 3+2 = 5$, $n+4 = 3+4 = 7$. The integers $n, n+2, n+4$ are prime numbers.
   (b) Take $x = \sqrt{2}$. By definition, $x \in \mathbb{R}$. Note that $x^2 - 2 = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.
   (c) Take $z = \dfrac{1+i}{\sqrt{2}}$. By definition, $z \in \mathbb{C}$.

   Note that $z^4 = \left(\dfrac{1+i}{\sqrt{2}}\right)^4 = \dfrac{1+4i+6i^2+4i^3+i^4}{4} = \dfrac{1+4i-6-4i+1}{4} = -1$.

   (d) Take $x = -\dfrac{1}{2}$. By definition, $x \in \mathbb{Q}$.

   Note that $(\log_2(-2x))^2 = \left(\log_2\left(-2 \cdot \left(-\dfrac{1}{2}\right)\right)\right)^2 = (\log_2(1))^2 = 0^2 = 0$.

   Also note that $-\log_2(4x^2) = -\log_2\left(4 \cdot \left(-\dfrac{1}{2}\right)^2\right) = -\log_2(1) = 0$.

   Then $(\log_2(-2x))^2 = -\log_2(4x^2)$.

4. **Answer.**

(a) i. There are exactly two elements in $A \cap B$. They are $\{0,1\}$, $\{1,2,3\}$.

ii. There are exactly five elements in $A \cup B$. They are $\{0,1\}$, $\{1\}$, $\{1,2,3\}$, $\{3,4\}$, $\{\{3\},\{4\}\}$.

iii. There are exactly two elements in $A \backslash B$. They are $\{1\}$, $\{3,4\}$.

iv. There is exactly one element in $B \backslash A$. It is $\{\{3\},\{4\}\}$.

v. There are exactly three elements in $A \triangle B$. They are $\{1\}$,$\{3,4\}$, $\{\{3\},\{4\}\}$.

vi. There are exactly four elements in $(A \cap B) \times (A \backslash B)$. They are $(\{0,1\},\{1\})$, $(\{1,2,3\},\{1\})$, $(\{0,1\},\{3,4\})$, $(\{1,2,3\},\{3,4\})$.

vii. There are exactly four elements in $\mathfrak{P}(A \backslash B)$. They are $\emptyset$, $\{\{1\}\}$, $\{\{3,4\}\}$, $\{\{1\},\{3,4\}\}$.

viii. There are exactly two elements in $\mathfrak{P}((\mathfrak{P}(B \backslash A) \backslash \{\emptyset\}))$. They are $\emptyset, \{B \backslash A\}$.

(b) i. There are exactly four elements in $C \cup D$. They are $\emptyset, \{\emptyset\}, \mathbb{N}, \mathbb{Z}, \{\mathbb{Q}\}$.

ii. There are exactly two elements in $D \cap E$. They are $\emptyset, \{\mathbb{Q}\}$.

iii. There are exactly two elements in $F \backslash E$. They are $\{\mathbb{N}, \mathbb{Z}\}, \{\mathbb{Z}, \mathbb{Q}\}$.

iv. There are exactly three elements in $(C \cup E) \backslash (D \cup F)$. They are $\{\emptyset\}, \{\mathbb{N}\}, \{\mathbb{Z}\}$.

v. There are exactly four elements in $\mathfrak{P}(C)$. They are $\emptyset, \{\{\emptyset\}\}, \{\mathbb{N}\}, C$.

vi. There are exactly four elements in $\mathfrak{P}(D) \cap \mathfrak{P}(E)$. They are $\emptyset$, $\{\emptyset\}, \{\{\mathbb{Q}\}\}, \{\emptyset, \{\mathbb{Q}\}\}$.

vii. There are exactly six elements in $C \times F$. They are $(\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\mathbb{N}, \mathbb{Z}\}), (\{\emptyset\}, \{\mathbb{Z}, \mathbb{Q}\}), (\mathbb{N}, \emptyset), (\mathbb{N}, \{\mathbb{N}, \mathbb{Z}\}), (\mathbb{N}, \{\mathbb{Z}, \mathbb{Q}\})$.

viii. There are exactly four elements in $(D \cap E)^2$. They are $(\emptyset, \emptyset), (\{\mathbb{Q}\}, \emptyset), (\emptyset, \{\mathbb{Q}\}), (\{\mathbb{Q}\}, \{\mathbb{Q}\})$.

5. **Answer.**

(a)
- $A$ is non-empty. Justification:—
  We verify that $40 \in A$:—
  Note that $40 \in \mathbb{N} \backslash \{0,1,2,3,4,5\}$.
  $40 = \dfrac{10^3}{5^2}$ and $10 \in \mathbb{N}$.
- $B = \emptyset$. Justification:—
  Suppose it were true that $B \neq \emptyset$.
  Pick some $x_0 \in B$. Then, $x_0 \in \mathbb{N} \backslash \{0,1,2,3,4,5\}$, and also, for any $k \in \mathbb{N}$, $x_0 = \dfrac{k^3}{5^2}$.
  Since $0 \in \mathbb{N}$, $x_0 = \dfrac{0^3}{5^2} = 0$.
  Recall that $x_0 \in \mathbb{N} \backslash \{0,1,2,3,4,5\}$. Then $x_0 \neq 0$.
  Contradiction arises.

(b)
- $C$ is non-empty. Justification:—
  We verify that $20 \in C$:—
  Note that $20 \in \mathbb{N} \backslash \{0,1,2,3,4,5\}$.
  $5^2 \cdot 20^2 = 10000 = (\sqrt[3]{1000})^4$ and $1000 \in \mathbb{N}$.
- $D = \emptyset$.
  Justification:—
  Suppose it were true that $D \neq \emptyset$.
  Pick some $x_0 \in D$. Then, $x_0 \in \mathbb{N} \backslash \{0,1,2,3,4,5\}$, and also, for any $k \in \mathbb{N}$, $5^2 {x_0}^2 = \left(\sqrt[3]{k}\right)^4$.
  Since $0 \in \mathbb{N}$, $5^2 {x_0}^2 = \left(\sqrt[3]{0}\right)^4 = 0$. Then $x_0 = 0$.
  Recall that $x_0 \in \mathbb{N} \backslash \{0,1,2,3,4,5\}$. Then $x_0 \neq 0$.
  Contradiction arises.

6. **Answer.**

(a) i. Suppose $A, B$ are sets. Then we say that $A$ is a subset of $B$ if the statement (†) holds:

(†) For any object $x$, if $x \in A$ then $x \in B$.

ii. (I) $C, D$

(II) there exists some object $x_0$ such that $x_0 \in C$ and $x_0 \notin D$

(b) i. (I) For any object $x$, if $x \in A$ then $x \in B$.

(II) $x \in A$

(III) there exists some

(IV) $\mathbb{Z}$

(V) $x = 16m^6$

(VI) $2m^2$

(VII) $2, m \in \mathbb{Z}$

(VIII) $\mathbb{Z}$

(IX) $2(8m^6) = 2(2m^2)^3 = 2n^3$

(X) $x \in B$

ii. (I) There exists some $x_0$ such that $x_0 \in B$ and $x_0 \notin A$.

(II) $x_0 = 2 \cdot 1^3$

(III) 1

(IV) $x_0 \in B$

(V) Suppose it were true that $x_0 \in A$.

(VI) there would exist some $m \in \mathbb{Z}$

(VII) $2 \cdot 4m^6$

(VIII) 1

*Alternative answer.* (VII) 1. (VIII) $2 \cdot 4m^6$

(IX) $4m^6 \in \mathbb{Z}$

(X) divisible by 2

(c) i. (I) For any $\zeta \in \mathbb{C}$, if $\zeta \in A$ then $\zeta \in B$.

(II) Suppose $\zeta \in A$

(III) $|\zeta| \leq 2$

(IV) $\leq$

(V) $(\mathsf{Im}(\zeta))^2$

(VI) $|\mathsf{Re}(\zeta)| \leq |\zeta|$

(VII) $|\mathsf{Re}(\zeta)| \leq 2$

(VIII) Note that $|\mathsf{Im}(\zeta)|^2 = (\mathsf{Im}(\zeta))^2 \leq (\mathsf{Re}(\zeta))^2 + (\mathsf{Im}(\zeta))^2 = |\zeta|^2$. Also note that $|\mathsf{Im}(\zeta)|$ and $|\zeta|$ are non-negative. Then $|\mathsf{Im}(\zeta)| \leq |\zeta|$. Therefore by $(\star)$, we have $|\mathsf{Im}(\zeta)| \leq 2$.

(IX) and

(X) $\zeta \in B$

ii. (I) There exists some $\zeta_0 \in \mathbb{C}$ such that $\zeta_0 \in B$ and $\zeta_0 \notin A$.

(II) Take $\zeta_0 = 2 + 2i$.

(III) $\mathsf{Re}(\zeta_0)$

(IV) $|\mathsf{Im}(\zeta_0)| \leq 2$

(V) $\zeta_0 \in B$

(VI) $|\zeta_0|^2$

(VII) 8

(VIII) 4

(IX) 2

(X) $\zeta_0 \notin A$

7. **Solution.**

(a) Let $A = \left\{\zeta \in \mathbb{C} : |\zeta - i| < 1\right\}$, $B = \left\{\zeta \in \mathbb{C} : |\zeta + i| < 3\right\}$.

[*Pictorial roughwork.* Give a sketch of $A, B$ on the Argand plane.
$A$ is the open disc with centre $i$ and radius 1.
$B$ is the open disc with centre $-i$ and radius 3.
The former lies entirely inside the latter.]

i. We verify $A \subset B$.

[*Reminder.* This amounts to proving '*for any $\zeta \in \mathbb{C}$, if $\zeta \in A$ then $\zeta \in B$*'.]

Pick any $\zeta \in \mathbb{C}$. Suppose $\zeta \in A$.
We have $|\zeta - i| < 1$ (by the definition of $A$).
By the Triangle Inequality, we have $|\zeta + i| = |\zeta - i + 2i| \leq |\zeta - i| + |2i| = |\zeta - i| + 2 < 1 + 2 = 3$.
Then $|\zeta + i| < 3$. Therefore, we have $\zeta \in B$ (by the definition of $B$).
It follows that $A \subset B$.

ii. We verify $B \not\subset A$.

[*Reminder.* This amounts to proving '*there exists some $\zeta_0 \in \mathbb{C}$ such that $\zeta_0 \in B$ and $\zeta_0 \notin A$*'.]

Take $\zeta_0 = 0$. By definition, $\zeta_0 \in \mathbb{C}$.
Note that $|\zeta_0 + i| = |0 + i| = 1 < 3$.
Then $\zeta_0 \in B$.
We verify that $\zeta_0 \notin A$:

- We have $|\zeta_0 - i| = |0 - i| = 1 \geq 1$.
  Then $\zeta_0 \notin A$.

It follows that $B \not\subset A$.

(b) Let $D = \left\{\zeta \in \mathbb{C} : |\zeta| \leq 5\right\}$, $E = \left\{\zeta \in \mathbb{C} : |\zeta - 4| + |\zeta + 4| \leq 10\right\}$, $F = \left\{\zeta \in \mathbb{C} : |\zeta| \leq 3\right\}$.

[*Pictorial roughwork.* Give a sketch of $D, E, F$ on the Argand plane.
$D$ is the closed disc with centre 0 and radius 5.
$E$ is the closed elliptical region with centre 0, foci at $4, -4$, vertices at $5, -5$ and covertices $3i, -3i$. (Its boundary is given by the equation $\dfrac{(\mathsf{Re}(z))^2}{25} + \dfrac{(\mathsf{Im}(z))^2}{9} = 1$ with complex unknown $z$.)
$F$ is the closed disc with centre 0 and radius 3.
It will appear that $F$ lies entirely inside $E$, and $E$ lies entirely inside $D$.]

i. We verify $D \not\subset E$.

[*Reminder.* This amounts to proving '*there exists some* $\zeta_0 \in \mathbb{C}$ *such that* $\zeta_0 \in D$ *and* $\zeta_0 \notin E$'.]

Take $\zeta_0 = 5i$. By definition, $\zeta_0 \in \mathbb{C}$.
We have $|\zeta_0| = 5 \leq 5$. Then $\zeta_0 \in D$.
We verify that $|\zeta_0| \notin E$:

- We have $|\zeta_0 - 4| + |\zeta_0 + 4| = |-4 + 5i| + |4 + 5i| = 2\sqrt{41} > 2\sqrt{36} = 12 > 10$.
  Then $\zeta_0 \notin E$.

It follows that $D \not\subset E$.

ii. We verify $E \subset D$.

[*Reminder.* This amounts to proving '*for any* $\zeta \in \mathbb{C}$, *if* $\zeta \in E$ *then* $\zeta \in D$'.]

Pick any $\zeta \in \mathbb{C}$. Suppose $\zeta \in E$.
We have $|\zeta - 4| + |\zeta + 4| \leq 10$.
By the Triangle Inequality, we have $2|\zeta| = |2\zeta| = |(\zeta - 4) + (\zeta + 4)| \leq |\zeta - 4| + |\zeta + 4| \leq 10$.
Then $|\zeta| \leq 5$.
Hence $\zeta \in D$.
It follows that $E \subset D$.

iii. We verify $E \not\subset F$.

[*Reminder.* This amounts to proving '*there exists some* $\zeta_0 \in \mathbb{C}$ *such that* $\zeta_0 \in E$ *and* $\zeta_0 \notin F$'.]

Take $\zeta_0 = 4$. By definition, $\zeta_0 \in \mathbb{C}$.
We have $|\zeta_0 - 4| + |\zeta_0 + 4| = 8 \leq 10$. Then $\zeta_0 \in E$.
We verify that $|\zeta_0| \notin F$:

- We have $|\zeta_0| = 4 > 3$.
  Then $\zeta_0 \notin F$.

It follows that $E \not\subset F$.

iv. We verify $F \subset E$.

[*Reminder.* This amounts to proving '*for any* $\zeta \in \mathbb{C}$, *if* $\zeta \in F$ *then* $\zeta \in E$'.]

Pick any $\zeta \in \mathbb{C}$. Suppose $\zeta \in F$.
We have $|\zeta| \leq 3$. Therefore $(\mathsf{Re}(\zeta))^2 + (\mathsf{Im}(\zeta))^2 = |\zeta|^2 \leq 9$.
Note that

$$\begin{array}{rcl}
(|\zeta - 4| + |\zeta + 4|)^2 & = & |\zeta - 4|^2 + |\zeta + 4|^2 + 2|\zeta - 4| \cdot |\zeta + 4| \\
& = & (\zeta - 4)(\bar{\zeta} - 4) + (\zeta + 4)(\bar{\zeta} + 4) + 2|(\zeta - 4)(\zeta + 4)| \\
& = & 2|\zeta|^2 + 32 + 2|\zeta^2 - 16| \\
& \leq & 2|\zeta|^2 + 32 + 2(|\zeta^2| + 16) \\
& = & 4|\zeta|^2 + 64 \\
& \leq & 4 \cdot 3^2 + 64 \\
& = & 100
\end{array}$$

Since $|\zeta - 4| + |\zeta + 4|$ and 100 are both non-negative, we have $|\zeta - 4| + |\zeta + 4| \leq 10$.
It follows that $F \subset E$.
**Remark.** The 'algbraic relation' '$(|z - 4| + |z + 4|)^2 = 2|z|^2 + 32 + 2|z^2 - 16|$' which holds for any arbitrary complex number $z$ is playing a crucial role in the argument. The inequality '$|z^2 - 16| \leq |z^2| + 16$' which holds for any arbitrary complex number $z$ is also crucial.

8. **Solution.**

(a) Let $A = \left\{ x \;\middle|\; \begin{array}{l} \text{There exists some } n \in \mathbb{Z} \\ \text{such that } 3x = 8n + 1 \end{array} \right\}$, $B = \left\{ x \;\middle|\; \begin{array}{l} \text{There exists some } n \in \mathbb{Z} \\ \text{such that } 9x = 4n - 1 \end{array} \right\}$.

[*Roughwork.* $A$ is the collection of rational numbers

$$\cdots, -5, -\frac{7}{3}, \frac{1}{3}, 3, \frac{17}{3}, \cdots.$$

$B$ is the collection of rational numbers

$$\cdots, -\frac{7}{3}, -\frac{17}{9}, -\frac{13}{9}, -1, -\frac{5}{9}, -\frac{1}{9}, \frac{1}{3}, \frac{7}{9}, \frac{11}{9}, \frac{5}{3}, \frac{19}{9}, \frac{23}{9}, 3, \cdots.$$

It seems that every element of $A$ belongs to $B$, and some element of $B$ does not belong to $A$.]

i. We verify $A \subset B$.

[*Reminder.* This amounts to proving '*for any $x$, if $x \in A$ then $x \in B$*'.]

Pick any object $x$. Suppose $x \in A$.
By the definition of $A$, there exists some $n \in \mathbb{Z}$ such that $3x = 8n + 1$.

[*Roughwork.* We want to name an appropriate $m$ which satisfies '$s \in \mathbb{Z}$' and '$9x = 4m - 1$' simultaneously. We ask: Does the equality '$3x = 8n + 1$' provide any hint?]

Take $m = 6n + 1$. Since $1, 6, n \in \mathbb{Z}$, we have $m \in \mathbb{Z}$.
We have $9x = 3 \cdot 3x = 3(8n + 1) = 24n + 3 = 4(6n + 1) - 1 = 4m - 1$.
Hence, by the definition of $B$, we have $x \in B$.
It follows that $A \subset B$.

ii. We verify $B \not\subset A$. [*Reminder.* This amounts to proving '*there exists some $x_0$ such that $x_0 \in B$ and $x_0 \notin A$*'.]

Let $x_0 = -\frac{1}{9}$.
Note that $9x_0 = -1 = 4 \cdot 0 - 1$ and $0 \in \mathbb{Z}$. Then $x_0 \in B$, by the definition of $B$.
We verify that $x_0 \notin A$ with the method of proof-by-contradiction:

- Suppose it were true that $x_0 \in A$.
  Then, by the definition of $A$, there would exist some $n \in \mathbb{Z}$ such that $3x_0 = 8n + 1$.
  Now $-\frac{1}{3} = 3 \cdot -\frac{1}{9} = 3x_0 = 8n + 1$.
  Since $1, 8, n \in \mathbb{Z}$, we have $8n + 1 \in \mathbb{Z}$. Then $-\frac{1}{3} \in \mathbb{Z}$.
  Note that $-\frac{1}{3}$ is not an integer.
  Contradiction arises.
  It follows that in the first place, $x_0 \notin A$.

Hence $A \not\subset B$.

(b) Let $C = \left\{ x \;\middle|\; \begin{array}{l} \text{There exist some } m, n \in \mathbb{Z} \\ \text{such that } x = 12m + 18n \end{array} \right\}$, $D = \left\{ x \;\middle|\; \begin{array}{l} \text{There exist some } m, n \in \mathbb{Z} \\ \text{such that } x = 6m + 8n \end{array} \right\}$.

[*Roughwork.* The elements of $C$ are necessarily integral multiples of 6.
Amongst the elements of $D$ is the integer 2 (because $2 = 6(-1) + 8 \cdot 1$). But then every even integer belongs to $D$.
So it seems that every element of $C$ belongs to $D$, but some element of $D$ does not belong to $C$.]

i. We verify $C \subset D$.

[*Reminder.* This amounts to proving '*for any $x$, if $x \in C$ then $x \in D$*'.]

Pick any $x$. Suppose $x \in C$.
Then, by the definition of $C$, there exists some $k, \ell \in \mathbb{Z}$ such that $x = 12k + 18\ell$.
We have $x = 6(2k + 3\ell) + 8 \cdot 0$.
Note that $0 \in \mathbb{Z}$.
Since $2, 3, k, \ell \in \mathbb{Z}$, we have $2k + 3\ell \in \mathbb{Z}$.
Then, by the definition $D$, we have $x \in D$.
It follows that $C \subset D$.

ii. We verify $D \not\subset C$.

[*Reminder.* This amounts to proving '*there exists some $x$ such that $x \in D$ and $x \notin C$*'.]

Let $x_0 = 2$.
Note that $x_0 = 6(-1) + 8 \cdot 1$ and $-1, 1 \in \mathbb{Z}$. Then $x_0 \in D$, by the definition of $D$.
We verify that $x_0 \notin C$ with the method of proof-by-contradiction:

- Suppose it were true that $x_0 \in C$.
  Then, by the definition of $C$, there would exist some $m, n \in \mathbb{Z}$ such that $x_0 = 12m + 18n$.
  Then $2 = x_0 = 12m + 18n = 6(2m + 3n)$. Therefore $2m + 3n = \frac{1}{3}$
  Since $2, 3, m, n \in \mathbb{Z}$, we would have $2m + 3n \in \mathbb{Z}$.
  Then $\frac{1}{3}$ would be an integer. But $\frac{1}{3}$ is not an integer.
  Contradiction arises.
  It follows that, in the first place, $x_0 \notin C$.

Hence $D \not\subset C$.

9. **Answer.**

(a) i. *Suppose $K, L$ be sets. Then we say $K$ is equal to $L$ as sets if both statements (†), (‡) hold:*

(†) *For any object $x$, if $x \in K$ then $x \in L$.*

(‡) *For any object $y$, if $y \in L$ then $y \in K$.*

*Alternative answer.*

*Suppose $K, L$ be sets. Then we say $K$ is equal to $L$ as sets if both statements $(\star)$, $(\star\star)$ hold:*

$(\star)$ *$K$ is a subset of $L$.*

$(\star\star)$ *$L$ is a subset of $K$.*

ii. The empty set is defined to be the set $\left\{x \,\middle|\, x \neq x\right\}$.

iii. Suppose $A, B$ are sets. Then the intersection of $A, B$ is defined to be the set $\left\{x \,\middle|\, x \in A \text{ and } x \in B\right\}$.

iv. Suppose $A, B$ are sets. Then the union of $A, B$ is defined to be the set $\left\{x \,\middle|\, x \in A \text{ or } x \in B\right\}$.

v. Suppose $A, B$ are sets. Then the complement of $B$ in $A$ is defined to be the set $\left\{x \,\middle|\, x \in A \text{ and } x \notin B\right\}$.

vi. Suppose $A$ is a set. Then the power set of $A$ is defined to be the set $\left\{S \,\middle|\, S \text{ is a subset of } A\right\}$.

(b) i. (I) $A \cup B \subset B$

(II) it were true that $A \backslash B \neq \emptyset$

(III) $x_0 \in A \backslash B$

(IV) $x_0 \in A$ and

(V) $x_0 \in A$

(VI) $x_0 \in B$

*Alternative answer.* (V) $x_0 \in B$ (VI) $x_0 \in A$

(VII) $x_0 \in A \cup B$

(VIII) $x_0 \in B$

(IX) and

(X) Contradiction arises

(XI) $A \backslash B = \emptyset$

ii. (I) Suppose $C, D$ are sets.

(II) Suppose $S \in \mathfrak{P}(C) \cup \mathfrak{P}(D)$

(III) $S \in \mathfrak{P}(C)$ or $S \in \mathfrak{P}(D)$

(IV) Suppose $S \in \mathfrak{P}(C)$.

(V) $S \subset C$

(VI) Since $x \in S$ and $S \subset C$, we have $x \in C$

(VII) $x \in C$ or $x \in D$

(VIII) $x \in C \cup D$

(IX) $S \subset C \cup D$

(X) $S \in \mathfrak{P}(C \cup D)$

(XI) Suppose $S \in \mathfrak{P}(D)$

(XII) $S \in \mathfrak{P}(C \cup D)$

iii. (I) if $x \in A \cap B$ then $x \in A$

(II) Suppose $x \in A \cap B$

(III) $x \in A$ and $x \in B$

(IV) $x \in A$

(V) For any object $x$, if $x \in A$ then $x \in A \cap B$

(VI) Pick any object $x$. Suppose $x \in A$.

(VII) $x \in A$ and $A \subset B$

(VIII) $x \in A$ and $x \in B$

(IX) $x \in A \cap B$

(X) $A \cap B \subset A$

(XI) $A \cap B = A$

(XII) For any object $x$, if $x \in A$ then $x \in B$

(XIII) $x \in A$

(XIV) $A \cap B = A$

(XV) by the definition of intersection, we have $x \in A$ and $x \in B$

(XVI) $x \in B$

iv. (I) Suppose $x \in C \backslash B$.

(II) complement

(III) $x \in C$ and $x \notin B$

(IV) Suppose it were true that $x \in A$.

(V) since $x \in A$ and

(VI) $x \in B$

(VII) and

(VIII) $x \in C$ and $x \notin A$

(IX) $x \in C \backslash A$

(X) Pick any object $x$. Suppose $x \in A$.

(XI) Suppose it were true that $x \notin B$.

(XII) and $A \subset C$

(XIII) $x \in C$

(XIV) and $x \notin B$

(XV) $x \in C \backslash B$

(XVI) $x \in C \backslash A$

(XVII) complement

(XVIII) $x \in C$ and $x \notin A$

(XIX) $x \notin A$

(XX) $x \in A$ and $x \notin A$

10. **Solution.**

(a) Let $A, B, C, D$ be sets. Suppose $A \subset C$ and $B \subset D$.

[We want to deduce $A \cup B \subset C \cup D$. According to definition, this is the same as deducing '*For any object $x$, if $x \in A \cup B$ then $x \in C \cup D$*'.]

Pick any object $x$.

Suppose $x \in A \cup B$. [We ask whether it is true that $x \in C \cup D$.]

By the definition of union, $x \in A$ or $x \in B$.

- (Case 1). Suppose $x \in A$. Then, since $A \subset C$, we have $x \in C$ by the definition of subset relation. Therefore $x \in C$ or $x \in D$. Hence $x \in C \cup D$ by the definition of union.
- (Case 2). Suppose $x \notin A$. Then $x \in B$. Therefore, since $B \subset D$, we have $x \in D$. Then $x \in C$ or $x \in D$. Hence $x \in C \cup D$.

Hence, in any case, we have $x \in C \cup D$.
It follows that $A \cup B \subset C \cup D$.

(b) Let $A, B, C$ be sets. Suppose $A \subset B$, $B \subset C$, and $C \subset A$.
By assumption, we have $A \subset B$. —— $(\star)$
We now verify that $B \subset A$:

- Pick any object $x$. Suppose $x \in B$.
  Then, since $x \in B$ and $B \subset C$, we have $x \in C$ (by the definition of subset relation).
  Since $x \in C$ and $C \subset A$, we have $x \in A$ (by the definition of subset relation).
  Hence it follows that $B \subset A$. —— $(\star\star)$

By $(\star)$ and $(\star\star)$ together, we have $A = B$ according to the definition of set equality.

(c) Let $A, B$ be sets. Suppose $A \subset A \backslash B$.
Further suppose it were true that $A \cap B \neq \emptyset$. Take some $x_0 \in A \cap B$. We have $x_0 \in A$ and $x_0 \in B$. In particular $x_0 \in A$. Also, $x_0 \in B$.
Since $x_0 \in A$ and $A \subset A \backslash B$, we would have $x_0 \in A \backslash B$. Then $x_0 \in A$ and $x_0 \notin B$ by definition of complement. In particular $x_0 \notin B$.
Now we have $x_0 \in B$ and $x_0 \notin B$. Contradiction arises.
It follows that $A \cap B = \emptyset$ in the first place.
*Alternative argument*:

Let $A, B$ be sets. Suppose $A \subset A \backslash B$.

- Suppose it were true that $A \cap B \neq \emptyset$. Take some $x_0 \in A \cap B$.
  We would have $x_0 \in A$ and $x_0 \in B$ by definition of complement. In particular $x_0 \in A$.
  Also, it would be false that $x_0 \notin B$. Then it would be false that ($x_0 \in A$ and $x_0 \notin B$). Therefore $x_0 \notin A \backslash B$.
  It would now follow that $A \not\subset A \backslash B$. Contradiction arises.

It follows that $A \cap B = \emptyset$.

(d) Let $A, B$ be sets. Suppose $A \cap B = \emptyset$.
Pick any object $x$. Suppose $x \in A$.
We claim that $x \notin B$. We justify this claim by applying the proof-by-contradiction method:

- Suppose it were true that $x \in B$.
  Recall that by assumption, $x \in A$ also.
  Then, we would have $x \in A$ and $x \in B$.
  Therefore, by the definition of intersection, $x \in A \cap B$.
  Recall that by assumption, $A \cap B = \emptyset$.
  Then $x \in \emptyset$. Contradiction arises.
  It follows that $x \notin B$ in the first place.

Now we have $x \in A$ and $x \notin B$.
Then by the definition of complement, $x \in A \backslash B$.
It follows that $A \subset A \backslash B$.

(e) Let $A, B, C$ be sets. Suppose $A \subset C$ and $B \subset C$.

- Pick any object $x$. Suppose $x \in (C \backslash A) \backslash (C \backslash B)$.
  Then $x \in C \backslash A$ and $x \notin C \backslash B$.
  In particular, $x \in C \backslash A$. Then $x \in C$ and $x \notin A$. In particular $x \notin A$.
  Recall that $x \notin C \backslash B$. It is not true that $x \in C \backslash B$. Then it is not true that ($x \in C$ and $x \notin B$). Therefore (it is not true that $x \in C$) or (it is not true that $x \notin B$). Hence $x \notin C$ or $x \in B$.
  Recall that $x \in C$ and $x \notin A$. In particular $x \in C$. It is impossible to have $x \notin C$. Therefore $x \in B$.
  Recall that $x \notin A$. Then $x \in B$ and $x \notin A$. Therefore $x \in B \backslash A$.
- Pick any object $x$. Suppose $x \in B \backslash A$. Then $x \in B$ and $x \notin A$. In particular $x \in B$. Since $B \subset C$, we have $x \in C$.
  Recall that $x \in B$ and $x \notin A$. In particular $x \notin A$. We have $x \in C$ and $x \notin A$. Then $x \in C \backslash A$.
  Recall that $x \in B$. Then $x \in B$ or $x \notin C$. Therefore (it is not true that $x \notin B$) or (it is not true that $x \in C$). Hence it is not true that ($x \notin B$ and $x \in C$). Then it is not true that $x \in C \backslash B$. Now we have $x \notin C \backslash B$.
  Therefore $x \in C \backslash A$ and $x \notin C \backslash B$. Hence $x \in (C \backslash A) \backslash (C \backslash B)$.

It follows that $(C\backslash A)\backslash(C\backslash B) = B\backslash A$.

(f) Let $A, B$ be sets. Suppose $\mathfrak{P}(B) \in \mathfrak{P}(A)$.
Pick any subset $S$ of $B$. We have $S \in \mathfrak{P}(B)$ by the definition of power set.
Since $\mathfrak{P}(B) \in \mathfrak{P}(A)$, we have $\mathfrak{P}(B) \subset A$ by the definition of power set.
Now we have $S \in \mathfrak{P}(B)$ and $\mathfrak{P}(B) \subset A$. Then $S \in A$ by the definition of subset relation.

11. **Solution.**

(a) i. The statement concerned is true. Justification:
- The elements of the set $\{1, 3, 5\}$ are $1, 3, 5$. Each of $1, 3, 5$ belongs to the set $\{1, 3, 5, 7\}$.
Therefore $\{1, 3, 5\}$ is a subset of $\{1, 3, 5, 7\}$.
Note that 7 belongs to the set $\{1, 3, 5, 7\}$, and 7 does not belong to the set $\{1, 3, 5\}$.
Then $\{1, 3, 5\}$ is not equal to $\{1, 3, 5, 7\}$.
Hence $\{1, 3, 5\}$ is a proper subset of $\{1, 3, 5, 7\}$.

ii. The statement concerned is false. Justification:
- Note that 9 belongs to $\{1, 3, 5, 9\}$ and 9 does not belong to $\{1, 3, 5, 7\}$.
Then $\{1, 3, 5, 9\}$ is not a subset of $\{1, 3, 5, 7\}$.
Therefore $\{1, 3, 5, 9\}$ is not a proper subset of $\{1, 3, 5, 7\}$.

iii. The statement concerned is false. Justification:
- The elements of the set $\{1, 1, 3, 5, 7\}$ are $1, 3, 5, 7$. Each of $1, 3, 5, 7$ belongs to the set $\{1, 3, 5, 7, 7\}$.
The elements of the set $\{1, 3, 5, 7, 7\}$ are $1, 3, 5, 7$. Each of $1, 3, 5, 7$ belongs to the set $\{1, 1, 3, 5, 7\}$.
Then $\{1, 3, 5, 7, 7\}$ is equal to $\{1, 1, 3, 5, 7\}$.
Therefore $\{1, 3, 5, 7, 7\}$ is not a proper subset of $\{1, 1, 3, 5, 7\}$.

(b) i. Suppose $A, B$ are sets.
- Suppose $A \subsetneqq B$. Then $A \subset B$ and $A \neq B$.
In particular $A \subset B$.
Since $A \neq B$, we have ($A \not\subset B$ or $B \not\subset A$).
Since $A \subset B$, we have $B \not\subset A$.
- Suppose $A \subset B$ and $B \not\subset A$.
Since $B \not\subset A$, it is not true that $B \subset A$. Therefore $A \neq B$.
It follows that $A \subsetneqq B$.

ii. Let $A, B, C$ be sets. Suppose $A \subset B$ and $B \subset C$. Further suppose $A \subsetneqq B$ or $B \subsetneqq C$.
Since $A \subset B$ and $B \subset C$, we have $A \subset C$.
We verify that $C \not\subset A$:
- Suppose it were true that $C \subset A$. Then, since $A \subset C$, we would have $A = C$. Moreover, since $A \subset B$ and $B \subset C = A$, we would have $A = B$.
Therefore $B = C$ also.
Now $A = B$ and $B = C$. Then ($A \subsetneqq B$ or $B \subsetneqq C$) would be false.
Contradiction arises. Hence $C \not\subset A$ in the first place.

Now we have $A \subset C$ and $C \not\subset A$. It follows that $A \subsetneqq C$.

12. **Answer.**

(a) *There are many correct answers for* (II), (III), ..., (IX) *collectively.*

(I) There exist some $x, y, z \in \mathbb{Z}$ such that each of $xy$, $xz$ is divisible by 4 and $xyz$ is not divisible by 8.
(II) $y = z = 1$
(III) 4
(IV) 4
(V) $4 = 1 \cdot 4$ and $1 \in \mathbb{Z}$
(VI) 4
(VII) 4 were divisible by 8
(VIII) $4 = 8k$
(IX) $\frac{1}{2}$

(b) (I) There exist some sets $A, B, C$ such that $A \cap B \neq \emptyset$ and $A \cap B \subset C$ and $A \not\subset C$ and $B \not\subset C$.
(II) $C = \{3\}$
(III) $\emptyset$
(IV) $A \cap B \subset C$
(V) and $1 \notin C$
(VI) $A \not\subset C$
(VII) $2 \in B$ and $2 \notin C$
(VIII) $B \not\subset C$

(c) (I) There exist some $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$ and $|x^2 - 2x| < |y^2 - 2y|$ and $x^2 > y^2$.
(II) $y = 1$
(III) $x > 0$ and $y > 0$

(IV) 0
(V) $|y^2 - 2y| = 1$
(VI) $|x^2 - x|$
(VII) $|y^2 - y|$
(VIII) $x^2 = 4$
(IX) $x^2 > y^2$

(d) (I) There exist some $m, n \in \mathbb{N}\backslash\{0, 1, 2\}$, $\zeta, \omega \in \mathbb{C}$ such that $m \neq n$ and $\zeta \neq \omega$ and $\zeta$ is an $m$-th root of unity and $\omega$ is an $n$-th root of unity and $\zeta\omega$ is not an $(m + n)$-th root of unity.
(II) Take
(III) $\omega = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$
(IV) $m \neq n$ and $\zeta \neq \omega$
(V) $\zeta$ is an $m$-th root of unity
(VI) $\omega^n = \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^8 = \cos\left(8 \cdot \frac{\pi}{4}\right) + i\sin\left(8 \cdot \frac{\pi}{4}\right) = \cos(2\pi) + i\sin(2\pi) = 1$
(VII) 12
(VIII) $(\zeta\omega)^{m+n} = \left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^{12} = \cos\left(12 \cdot \frac{3\pi}{4}\right) + i\sin\left(12 \cdot \frac{3\pi}{4}\right) = \cos(9\pi) + i\sin(9\pi) = -1$
(IX) $\neq$
(X) $\zeta\omega$ is not an $(m + n)$-th root of unity

(e) (I) Suppose
(II) $u \in \mathbb{R}\backslash\{-1, 0, 1\}$
(III) $u^6 + v^6 \leq 2v^4$
(IV) $u^6 - 2u^4 + u^2 + v^6 - 2v^4 + v^2 \leq 0$
(V) $v^2(v^2 - 1)$
(VI) $v^2(v^2 - 1)^2 = 0$
(VII) $u^2(u^2 - 1)^2 = 0$
(VIII) $u \in \mathbb{R}\backslash\{-1, 0, 1\}$

(f) (I) Suppose there existed some $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that $\zeta$ was both an 89-th root of unity and a 55-th root of unity.
(II) 1
(III) $\zeta^{89} = 1$
(IV) $\zeta^{21} = \zeta^{55}/\zeta^{34} = 1$, $\zeta^{13} = \zeta^{34}/\zeta^{21} = 1$, $\zeta^8 = \zeta^{21}/\zeta^{13} = 1$, $\zeta^5 = \zeta^{13}/\zeta^8 = 1$, $\zeta^3 = \zeta^8/\zeta^5 = 1$, $\zeta^2 = \zeta^5/\zeta^3 = 1$, $\zeta = \zeta^3/\zeta^2 = 1$.
(V) $\mathbb{C}\backslash\mathbb{R}$
(VI) and

13. **Solution.**

(a) Denote by $M$ the statement below:

$M$: Let $x, y, z \in \mathbb{N}$. Suppose $x + y$, $y + z$ are divisible by 3. Then $x + z$ is divisible by 3.

The negation of $M$ reads:

$\sim M$: There exist some $x, y, z \in \mathbb{N}$ such that $x + y$, $y + z$ are divisible by 3 and $x + z$ is not divisible by 3.

We verify $\sim M$:

- Take $x = z = 1$, $y = 2$.
  We have $x, y, z \in \mathbb{N}$.
  Note that $x + y = y + z = 3 = 1 \cdot 3$. We have $1 \in \mathbb{Z}$.
  Then, by definition, $x + y$, $y + z$ are divisible by 3.
  Note that $x + z = 2$. We verify that 2 is not divisible by 3:
  * Suppose 2 were divisible by 3.
    Then there would exist some $k \in \mathbb{Z}$ such that $2 = 3k$.
    For the same $k$, we would have $k = \frac{2}{3}$. Then $k$ is not an integer.
    Contradiction arises.

(b) Denote by $M$ the statement below:

$M$: Let $x, y, z \in \mathbb{N}$. Suppose $x - y > 0$ and $x - z > 0$ and $x - z$, $y - z$ are divisible by 5. Then $x + y + z$ is not divisible by 5.

The negation of $M$ reads:

$\sim M$: There exist some $x, y, z \in \mathbb{N}$ such that $x - y > 0$ and $x - z > 0$ and $x - z$, $y - z$ are divisible by 5 and $x + y + z$ is divisible by 5.

We verify $\sim M$:

- Take $x = y = 10$, and $z = 5$.
  We have $x, y, z \in \mathbb{N}$.
  Note that $x - z = y - z = 5 > 0$.
  Also note that $5 = 1 \cdot 5$, and $1 \in \mathbb{Z}$. Then $x - z$, $y - z$ are divisible by 5.
  Note that $x + y + z = 25$, and $25 = 5 \cdot 5$ and $5 \in \mathbb{Z}$. Then $x + y + z$ is divisible by 5.

(c) Denote by $M$ the statement below:

$M$: Suppose $x, y \in \mathbb{N}$. Then $\sqrt{x^2 + y^2} \in \mathbb{N}$.

The negation of $M$ reads:

$\sim M$: There exist some $x, y \in \mathbb{N}$ such that $\sqrt{x^2+y^2} \notin \mathbb{N}$.

We verify $\sim M$:

- Take $x = 1$, $y = 2$.
  Note that $x, y \in \mathbb{N}$.
  We have $x^2 + y^2 = 5$. Then $\sqrt{x^2+y^2} = \sqrt{5} \notin \mathbb{N}$.

(d) Denote by $M$ the statement below:

$M$: For any $s, t \in \mathbb{R}$, if both of $s + t, st$ are rational, then at least one of $s, t$ is rational.

The negation of $M$ reads:

$\sim M$: There exist some $s, t \in \mathbb{R}$ such that both of $s + t, st$ are rational and both of $s, t$ are irrational.

We verify $\sim M$:

- Take $s = \sqrt{2}$, $t = -\sqrt{2}$.
  Note that $s, t \in \mathbb{R}$. Both of $s, t$ are irrational numbers.
  We have $s + t = 0$ and $st = -2$.
  Then both of $s + t, st$ are rational.

(e) Denote by $M$ the statement below:

$M$: For any $a, b, c \in \mathbb{N}$, if $ab$ is divisible by $c$ and $c < a$ and $c < b$, then at least one of $a, b$ is divisible by $c$.

The negation of $M$ reads:

$\sim M$: There exist some $a, b, c \in \mathbb{N}$ such that $ab$ is divisible by $c$ and $c < a$ and $c < b$ and each of $a, b$ is not divisible by $c$.

We verify $\sim M$:

- Take $a = 8$, $b = 9$, $c = 6$. Note that $a, b, c \in \mathbb{N}$, and $c < a$ and $c < b$.
  We have $ab = 72 = 12 \cdot 6$ and $12 \in \mathbb{Z}$. Then $ab$ is divisible by $c$.
  Note that $a$ is not divisible by $c$, and $b$ is not divisible by $c$. (Fill in the detail.)

(f) Denote by $M$ the statement below:

$M$: Let $n$ be a positive integer, and $\zeta$ be a complex number. Suppose $\zeta$ is an $n^2$-th root of unity. Then $\zeta^2$ is an $n$-th root of unity.

The negation of $M$ reads:

$\sim M$: There exist some positive integer $n$ and some complex number $\zeta$ such that $\zeta$ is an $n^2$-th root of unity and $\zeta^2$ is not an $n$-th root of unity.

We verify $\sim M$:

- Take $n = 3$, $\zeta = \cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)$.
  $\zeta^{n^2} = \zeta^{3^2} = \zeta^9 = \cos\left(9 \cdot \frac{2\pi}{9}\right) + i\sin\left(9 \cdot \frac{2\pi}{9}\right) = \cos(2\pi) + i\sin(2\pi) = 1$.
  Then $\zeta$ is a $n^2$-th root of unity.
  $\zeta^2 = \cos\left(2 \cdot \frac{2\pi}{9}\right) + i\sin\left(2 \cdot \frac{2\pi}{9}\right) = \cos\left(\frac{4\pi}{9}\right) + i\sin\left(\frac{4\pi}{9}\right)$.
  $(\zeta^2)^n = (\zeta^2)^3 = \cos\left(3 \cdot \frac{4\pi}{9}\right) + i\sin\left(3 \cdot \frac{4\pi}{9}\right) = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \neq 1$.
  Then $\zeta^2$ is not an $n$-th root of unity.

(g) Denote by $M$ the statement below:

$M$: Let $n$ be a positive integer, and $\zeta$ be a complex number. Suppose $\zeta^n$ is an $n$-th root of unity. Then $\zeta$ is a $(2n)$-th root of unity.

The negation of $M$ reads:

$\sim M$: There exist some positive integer $n$ and some complex number $\zeta$ such that $\zeta^n$ is an $n$-th root of unity and $\zeta$ is not a $(2n)$-th root of unity.

We verify $\sim M$:

- Take $n = 3$, $\zeta = \cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)$.
  Note that $\zeta^n = \zeta^3 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$.

Then $(\zeta^n)^n = (\zeta^3)^3 = \cos(2\pi) + i\sin(2\pi) = 1$.
Therefore $\zeta^n$ is a $n$-th root of unity.
Note that $\zeta^{2n} = \cos\left(6 \cdot \frac{2\pi}{9}\right) + i\sin\left(6 \cdot \frac{2\pi}{9}\right) = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \neq 1$. Then $\zeta$ is not a $(2n)$-th root of unity.

14. **Solution.**

(a) Denote by $M$ the statement below:

$M$: Suppose $A, B, C$ are sets. Then $A \backslash (C \backslash B) \subset A \cap B$.

The negation of $M$ reads:

$\sim M$: There exist some sets $A, B, C$ such that $A \backslash (C \backslash B) \not\subset A \cap B$.

We verify $\sim M$:

- Regard $0, 1, 2$ as distinct objects.
  Let $A = \{0, 1\}$, $B = \{1\}$, $C = \{2\}$.
  We have $A \cap B = B = \{1\}$, $C \backslash B = C = \{2\}$, $A \backslash (C \backslash B) = A = \{0, 1\}$.
  Note that $0 \in A \backslash (C \backslash B)$ and $0 \notin A \cap B$.
  Hence $A \backslash (C \backslash B) \not\subset A \cap B$.

(b) Denote by $M$ the statement below:

$M$: Suppose $A, B, C$ be non-empty sets. Then $B \backslash A \subset (C \backslash A) \backslash (C \backslash B)$.

The negation of $M$ reads:

$\sim M$: There exist some sets $A, B, C$ such that each of $A, B, C$ is non-empty and $B \backslash A \not\subset (C \backslash A) \backslash (C \backslash B)$.

We verify $\sim M$:

- Regard $0, 1, 2$ as distinct objects.
  Let $A = \{0\}$, $B = \{1\}$, $C = \{2\}$.
  $A, B, C$ are non-empty sets.
  We have $B \backslash A = B = \{1\}$, $C \backslash A = C = \{2\}$, $C \backslash B = C = \{2\}$, and $(C \backslash A) \backslash (C \backslash B) = \emptyset$.
  Note that $1 \in B \backslash A$ and $1 \notin (C \backslash A) \backslash (C \backslash B)$.
  Hence $B \backslash A \not\subset (C \backslash A) \backslash (C \backslash B)$.

(c) Denote by $M$ the statement below:

$M$: Suppose $A, B, C$ are non-empty sets. Then $A \cup (B \cap C) \subset (A \cup B) \cap C$.

The negation of $M$ reads:

$\sim M$: There exists some sets $A, B, C$ such that each of $A, B, C$ is non-empty and $A \cup (B \cap C) \not\subset (A \cup B) \cap C$.

We verify $\sim M$:

- Regard $0, 1, 2$ as distinct objects.
  Let $A = \{0\}, B = \{1\}, C = \{2\}$.
  We have $B \cap C = \emptyset$. Then $A \cup (B \cap C) = \{0\}$.
  We also have $A \cup B = \{0, 1\}$. Then $(A \cup B) \cap C = \emptyset$.
  Note that $0 \in A \cup (B \cap C)$ and $0 \neq (A \cup B) \cap C$
  Here $A \cup (B \cap C) \not\subset (A \cup B) \cap C$.

(d) Denote by $M$ the statement below:

$M$: Suppose $A, B, C$ are non-empty sets. Then $B \cap C \subset [A \backslash (B \backslash C)] \cup [B \backslash (C \backslash A)]$.

The negation of $M$ reads:

$\sim M$: There exist some sets $A, B, C$ such that each of $A, B, C$ is non-empty and $B \cap C \not\subset [A \backslash (B \backslash C)] \cup [B \backslash (C \backslash A)]$.

We verify $\sim M$:

- Regard $0, 1$ as distinct objects.
  Let $A = \{0\}$ and $B = C = \{0, 1\}$.
  We have $B \cap C = \{0, 1\}$.
  We also have $B \backslash C = \emptyset$, $A \backslash (B \backslash C) = \{0\}$.
  Moreover $C \backslash A = \{1\}$, $B \backslash (C \backslash A) = \{0\}$.
  Then we have $[A \backslash (B \backslash C)] \cup [B \backslash (C \backslash A)] = \{0\}$.
  Note that $1 \in B \cap C$ and $1 \notin [A \backslash (B \backslash C)] \cup [B \backslash (C \backslash A)]$.
  Hence $B \cap C \not\subset [A \backslash (B \backslash C)] \cup [B \backslash (C \backslash A)]$.

(e) Denote by $M$ the statement below:

$M$: Let $A, B, C$ be sets. Suppose $A \cap B \subset C$. Then $C \subset (A \cap C) \cup (B \cap C)$.

The negation of $M$ reads:

$\sim M$: There exist some sets $A, B, C$ such that $A \cap B \subset C$ and $C \not\subset (A \cap C) \cup (B \cap C)$.

We verify $\sim M$:

- Regard $0, 1, 2$ as distinct objects.
  Take $A = \{1\}, B = \{2\}, C = \{0, 1, 2\}$. We have $A \cap B = \emptyset \subset C$.
  Note that $A \cap C = A = \{1\}$ and $B \cap C = B = \{2\}$. Then $(A \cap C) \cup (B \cap C) = \{1, 2\}$.
  $0 \in C$ and $0 \notin (A \cap C) \cup (B \cap C)$.
  Hence $C \not\subset (A \cap C) \cup (B \cap C)$.

(f) Denote by $M$ the statement below:

$M$: Let $A, B, C$ be sets. Suppose $A\backslash B$, $A\backslash C$ are non-empty. Then $A\backslash(B \cap C) \subset (A\backslash B) \cap (A\backslash C)$.

The negation of $M$ reads:

$\sim M$: There exist some sets $A, B, C$ such that $A\backslash B$, $A\backslash C$ are non-empty and $A\backslash(B \cap C) \not\subset (A\backslash B) \cap (A\backslash C)$.

We verify $\sim M$:

- Regard $0, 1, 2$ as distinct objects.
  Take $A = \{0, 2\}$, $B = \{1\}$, $C = \{1, 2\}$.
  We have $A\backslash B = \{0, 2\} \neq \emptyset$ and $A\backslash C = \{1\} \neq \emptyset$.
  We have $B \cap C = \{1\}$, $A\backslash(B \cap C) = \{0, 2\}$.
  We have $(A\backslash B) \cap (A\backslash C) = \emptyset$.
  Note that $0 \in A\backslash(B \cap C)$ and $0 \notin (A\backslash B) \cup (A\backslash C)$.
  Then $A\backslash(B \cap C) \not\subset (A\backslash B) \cap (A\backslash C)$.

(g) Denote by $M$ the statement below:

$M$: Let $A, B, C, D$ be non-empty sets. Suppose $A \subset C$ and $B \subset D$. Further suppose $C \cap D \neq \emptyset$. Then $A \cup B \subset C \cap D$.

The negation of $M$ reads:

$\sim M$: There exist some sets $A, B, C, D$ such that each of $A, B, C, D$ is non-empty and $A \subset C$ and $B \subset D$ and $C \cap D \neq \emptyset$ and $A \cup B \not\subset C \cap D$.

We verify $\sim M$:

- Regard $0, 1, 2$ as distinct objects.
  Take $A = \{1\}$, $B = \{2\}$, $C = \{0, 1\}$, $D = \{0, 2\}$.
  $A, B, C, D$ are all non-empty sets. $A \subset C$ and $B \subset D$.
  $C \cap D = \{0\}$. Then $C \cap D \neq \emptyset$.
  $A \cup B = \{1, 2\}$. Note that $1 \in A \cup B$ and $1 \notin C \cap D$. Then $A \cup B \not\subset C \cap D$.

15. **Solution.**

(a) *Method (A).*

Denote by $N$ the statement below:

$N$: There exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.

The negation of $N$ reads:

$\sim N$: For any $x \in \mathbb{R}$, $x^2 + 2x + 3 \geq 0$.

We verify $\sim N$:

- Pick any $x \in \mathbb{R}$.
  We have $x^2 + 2x + 3 = (x + 1)^2 + 2$. —— $(\star)$
  Since $x \in \mathbb{R}$, we have $x + 1 \in \mathbb{R}$. Then $(x + 1)^2 \geq 0$.
  Therefore by $(\star)$, we have $x^2 + 2x + 3 \geq 0 + 2 = 2 \geq 0$.

*Method (B).*

[Denote by $N$ the statement below:

$N$: *There exists some* $x \in \mathbb{R}$ *such that* $x^2 + 2x + 3 < 0$.

We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.
Note that $x^2 + 2x + 3 = (x + 1)^2 + 2$. —— $(\star)$
Since $x \in \mathbb{R}$, we would have $x + 1 \in \mathbb{R}$. Then $(x + 1)^2 \geq 0$.
By $(\star)$, we would have $x^2 + 2x + 3 \geq 0 + 2 = 2 \geq 0$.
Then $0 \leq x^2 + 2x + 3 < 0$. Contradiction arises.
Hence, in the first place, it is false that there exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.

(b) *Method (A).*

Denote by $N$ the statement below:

$N$: There exist some $x, y \in \mathbb{R}\backslash\{0\}$ such that $(x + y)^2 = x^2 + y^2$.

The negation of $N$ reads:

$\sim N$: For any $x, y \in \mathbb{R}\backslash\{0\}$, $(x + y)^2 \neq x^2 + y^2$.

We verify $\sim N$:

- Pick any $x, y \in \mathbb{R}\backslash\{0\}$.
  We have $xy \neq 0$. Then $(x + y)^2 - x^2 - y^2 = 2xy \neq 0$.
  Therefore $(x + y)^2 \neq x^2 + y^2$.

*Method (B).*

[Denote by $N$ the statement below:
$N$: *There exist some* $x, y \in \mathbb{R}\backslash\{0\}$ *such that* $(x + y)^2 = x^2 + y^2$.
We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $x, y \in \mathbb{R}\backslash\{0\}$ such that $(x + y)^2 = x^2 + y^2$.
Then we would have $2xy = (x + y)^2 - x^2 - y^2 = 0$.
Since $x \neq 0$ and $y \neq 0$ and $2 \neq 0$, we have $2xy \neq 0$.
Contradiction arises.

(c) *Method (A).*

Denote by $N$ the statement below:

$N$: *There exists some* $r \in \mathbb{R}$ *such that* $r < r^5 \leq r^3$.

We may re-formulation $N$ as:

$N$: *There exists some* $r \in \mathbb{R}$ *such that* $r < r^5$ *and* $r^5 \leq r^3$.

One formulation of the negation of $N$ reads:

$\sim N$: For any $r \in \mathbb{R}$, if $r < r^5$ then $r^5 > r^3$.

We verify $\sim N$:

Pick any $r \in \mathbb{R}$. Suppose $r < r^5$.
Then $r(r^2 - 1)(r^2 + 1) = r^5 - r > 0$. —— $(\sharp)$
Since $r$ is a real number, $r^2 \geq 0$. Then $r^2 + 1 > 0$.
By $(\sharp)$, we have $r(r^2 - 1) > 0$. —— $(\natural)$
Note that $r \neq 0$; otherwise we would have $r(r^2 - 1) = 0$. Then $r^2 > 0$.
Therefore by $(\natural)$, we have $r^5 - r^3 = r^2 \cdot r(r^2 - 1) > 0$.
Hence $r^5 > r^3$.

*Method (B).*

[Denote by $N$ the statement below:
$N$: *There exists some* $r \in \mathbb{R}$ *such that* $r < r^5 \leq r^3$.
We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $r \in \mathbb{R}$ such that $r < r^5 \leq r^3$.
We would have $r < r^3$.
Note that $r \neq 0$; otherwise we would have $r = r^3 = r^5 = 0$. Since $r$ is a real number, $r^2 > 0$.
Then, since $r^5 \leq r^3$, we would have $r^3 = \dfrac{r^5}{r^2} \leq \dfrac{r^3}{r^2} = r$.
Now we would have $r < r^3$ and $r^3 \leq r$. Therefore $r < r$. Contradiction arises.

(d) *Method (A).*

Denote by $N$ the statement below:

$N$: *There exists some $\zeta \in \mathbb{C}\backslash\{1\}$, $n \in \mathbb{N}\backslash\mathbb{N}\backslash\{0,1\}$ such that $\zeta$ is an $(n+1)$-th root of unity and $\zeta$ is an $(n^2+n+1)$-th root of unity.*

One formulation of the negation of $N$ reads:

$\sim N$: For any $\zeta \in \mathbb{C}\backslash\{1\}$, for any $n\backslash\{0,1\}$, if $\zeta$ is an $(n+1)$-th root of unity then $\zeta$ is not an $(n^2+n+1)$-th root of unity.

We verify $\sim N$:

Pick any $\zeta \in \mathbb{C}\backslash\{1\}$, $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $\zeta$ is an $(n+1)$-th root of unity.

[We want to deduce that $\zeta$ is not an $(n^2+n+1)$-th root of unity.]

By assumption, we have $\zeta^{n+1} = 1$.
Then $\zeta^{n^2+n+1} = \zeta^{(n+1)n+1} = (\zeta^{n+1})^n \cdot \zeta = 1^n \cdot \zeta = \zeta \neq 1$.
Therefore $\zeta$ is not an $(n^2+n+1)$-th root of unity.

*Method (B).*

[Denote by $N$ the statement below:

$N$: *There exists some $\zeta \in \mathbb{C}\backslash\{1\}$, $n \in \mathbb{N}\backslash\{0,1\}$ such that $\zeta$ is an $(n+1)$-th root of unity and $\zeta$ is an $(n^2+n+1)$-th root of unity.*

We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $\zeta \in \mathbb{C}\backslash\{1\}$, $n \in \mathbb{N}\backslash\{0,1\}$ such that $\zeta$ was an $(n+1)$-th root of unity and $\zeta$ was an $(n^2+n+1)$-th root of unity.

By assumption, we would have $\zeta^{n+1} = 1$ and $\zeta^{n^2+n+1} = 1$.

Then $1 = \zeta^{n^2+n+1} = \zeta^{(n+1)n+1} = (\zeta^{n+1})^n \cdot \zeta = 1^n \cdot \zeta = \zeta$.

But $\zeta \neq 1$ by assumption. Contradiction arises.

(e) *Method (A).*

Denote by $N$ the statement below:

$N$: There exists some $s \in \mathbb{Q}$ such that (for any $t \in \mathbb{Q}$, $s = 2t+1$).

The negation of $N$ reads:

$\sim N$: For any $s \in \mathbb{Q}$, there exists some $t \in \mathbb{Q}$ such that $s = 2t+1$.

We verify $\sim N$:

- Pick any $s \in \mathbb{Q}$.
  Define $t = \dfrac{s-1}{2}$. Since $s, 1, 2 \in \mathbb{Q}$, we have $t \in \mathbb{Q}$.
  By definition, we have $2t+1 = 2 \cdot \dfrac{s-1}{2} + 1 = (s-1)+1 = s$.

*Method (B).*

[Denote by $N$ the statement below:

$N$: *There exists some $s \in \mathbb{Q}$ such that (for any $t \in \mathbb{Q}$, $s = 2t+1$).*

We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $s \in \mathbb{Q}$ such that (for any $t \in \mathbb{Q}$, $s = 2t+1$).
Note that $0 \in \mathbb{Q}$. Then, for the same $s$, we would have $s = 2 \cdot 0 + 1 = 1$.
Also note that $1 \in \mathbb{Q}$. Then, for the same $s$, we would have $s = 2 \cdot 1 + 1 = 3$.
Then $1 = 3$. Contradiction arises.
Hence, in the first place, it is false that there exists some $s \in \mathbb{Q}$ such that (for any $t \in \mathbb{Q}$, $s = 2t+1$).

(f) *Method (A).*

Denote by $N$ the statement below:

$N$: *There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).*

The negation of $N$ reads:

$\sim N$: For any $t \in \mathbb{R}$, there exists some $s \in \mathbb{C}$ such that $|s| > t$.

We verify $\sim N$:

- Pick any $t \in \mathbb{R}$.
  Take $s = |t| + 1$. By definition, $s \in \mathbb{C}$.
  Note that $s$ is a positive real number. Then $|s| = ||t|+1| = |t|+1 > |t| \geq t$.

*Method (B).*

[Denote by $N$ the statement below:

$N$: There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).

We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).

For this real number $t$, the statement '*for any* $s \in \mathbb{C}$, $|s| \leq t$' would be true.

Note that $|t| + 1$ is a complex number.

Then $||t| + 1| \leq t$.

Since $|t| + 1$ is a non-negative real number, we have $||t| + 1| = |t| + 1$.

Then we have $|t| + 1 \leq t \leq |t|$. Therefore $1 \leq 0$.

Contradiction arises.

(g) *Method (A).*

Denote by $N$ the statement below:

$N$: *There exist some* $a \in \mathbb{R}$, $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$ *such that* $\dfrac{(1 + \sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} \leq \dfrac{a^2}{24}$.

The negation of $N$ reads:

$\sim N$: For any $a \in \mathbb{R}$, for any $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$, $\dfrac{(1 + \sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} > \dfrac{a^2}{24}$.

We verify $\sim N$:

- Pick any $a \in \mathbb{R}$. Pick any $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$.
  By the Binomial Theorem,

$$\begin{aligned}
\left(1 + \sqrt{|a|}\right)^n &= \sum_{j=0}^{n} \binom{n}{j} \cdot (\sqrt{|a|})^j \\
&\geq 1 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\sqrt{|a|}\right)^4 \\
&> \frac{n(n-1)(n-2)(n-3)}{24} a^2.
\end{aligned}$$

  (The first inequality holds because $\binom{n}{j} \cdot (\sqrt{|a|})^j \geq 0$ for each $j$. The second holds because $1 > 0$.)

  Since $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$, we have $n(n-1)(n-2)(n-3) > 0$.

  Then $\dfrac{(1 + \sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} > \dfrac{a^2}{24}$.

*Method (B).*

[Denote by $N$ the statement below:

$N$: There exist some $a \in \mathbb{R}$, $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$ such that $\dfrac{(1 + \sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} \leq \dfrac{a^2}{24}$.

We dis-prove the statement $N$ by obtaining a contradiction from it.]

Suppose it were true that there existed some $a \in \mathbb{R}$, $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$ such that

$$\frac{\left(1 + \sqrt{|a|}\right)^n}{n(n-1)(n-2)(n-3)} \leq \frac{a^2}{24}.$$

By the Binomial Theorem,

$$\begin{aligned}
\left(1 + \sqrt{|a|}\right)^n &= \sum_{j=0}^{n} \binom{n}{j} \cdot (\sqrt{|a|})^j \\
&\geq 1 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\sqrt{|a|}\right)^4 \\
&> \frac{n(n-1)(n-2)(n-3)}{24} a^2.
\end{aligned}$$

Since $n \in \mathbb{N}\backslash\{0,1,2,3\}$, we have $n(n-1)(n-2)(n-3) > 0$.

Then $\dfrac{(1+\sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} > \dfrac{a^2}{24}$.

Now, by assumption, $\dfrac{a^2}{24} \geq \dfrac{(1+\sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} > \dfrac{a^2}{24}$.

Contradiction arises.

16. **Solution.**

(a) *Method (A).*

Denote by $N$ the statement below:

$N$: There exists some $x \in \mathbb{R}$ such that $|x+1| > |x|+1$.

The negation of $N$ reads:

$\sim N$: For any $x \in \mathbb{R}$, $|x+1| \leq |x|+1$.

We verify $\sim N$:

- Pick any $x \in \mathbb{R}$. We have $x < -1$ or $-1 \leq x \leq 0$ or $x > 0$.

  (Case 1). Suppose $x < -1$. Then $x+1 < 0$ and $x < 0$. We have $|x+1| = -(x+1) = -x-1 = |x|-1 \leq |x|+1$.

  (Case 2). Suppose $-1 \leq x \leq 0$. Then $x+1 \geq 0$ also. We have $|x+1| = x+1 \leq 0+1 = 1 \leq |x|+1$.

  (Case 3). Suppose $x > 0$. Then $x+1 > 0$ also. We have $|x+1| = x+1 = |x|+1 \leq |x|+1$.

  Hence, in any case, we have $|x+1| \leq |x|+1$.

*Alternative argument with Method (A).*

Denote by $N$ the statement below:

$N$: There exists some $x \in \mathbb{R}$ such that $|x+1| > |x|+1$.

The negation of $N$ reads:

$\sim N$: For any $x \in \mathbb{R}$, $|x+1| \leq |x|+1$.

We verify $\sim N$:

Pick any $x \in \mathbb{R}$. Suppose it were true that $|x+1| > |x|+1$ for this $x$.
Note that $|x+1| > |x|+1 \geq 1 > 0$.
Then $x^2+2x+1 = (x+1)^2 = |x+1|^2 > (|x|+1)^2 = x^2+2|x|+1$.
Therefore $x > |x| \geq x$.
Contradiction arises.
Hence $|x+1| \leq |x|+1$.

*Method (B).*

Suppose it were true that there existed some $x \in \mathbb{R}$ such that $|x+1| > |x|+1$.

Note that $|x+1| > |x|+1 \geq 1 > 0$.

Then $x^2+2x+1 = (x+1)^2 = |x+1|^2 > (|x|+1)^2 = x^2+2|x|+1$.

Then $x > |x| \geq x$.

Contradiction arises.

(b) Suppose it were true that there existed some $z \in \mathbb{C}$ such that $|z+3-4i| > |z|+5$.

Note that $|z|+5 \geq 0$.

Then

$$\begin{aligned} |z|^2+10|z|+25 &= (|z|+5)^2 \\ &< |z+3-4i|^2 \\ &= (z+3-4i)(\bar{z}+3+4i) \\ &= |z|^2+(3+4i)z+(3-4i)\bar{z}+25 \\ &= |z|^2+2\mathsf{Re}((3+4i)z)+25. \end{aligned}$$

Therefore $10|z| < 2\mathsf{Re}((3+4i)z) \leq 2|(3+4i)z| = 2|3+4i||z| = 10|z|$. Contradiction arises.

**Remark.** We may simply quote the Triangle Inequality in the argument:

Suppose it were true that there existed some $z \in \mathbb{R}$ such that $|z+3-4i| > |z|+5$.
By Triangle Inequality, we have $|z+3-4i| \leq |z|+|3-4i| = |z|+5$.
Then $|z+3-4i| \leq |z|+5 < |z+3-4i|$. Contradiction arises.

(c) Suppose it were true that there existed some $x \in \mathbb{R}$ such that $|x+4| > 2|x+1| + |x-2|$.
Then

$$\begin{aligned} 4|x+1|^2 + |x-2|^2 + 4|x+1||x-2| &= (2|x+1| + |x-2|)^2 \\ &< |x+4|^2 \\ &= (x+4)^2 \\ &= [2(x+1) + (-x+2)]^2 \\ &= 4(x+1)^2 + (2-x)^2 + 4(x+1)(-x+2) \\ &= 4|x+1|^2 + |x-2|^2 + 4(x+1)(-x+2) \end{aligned}$$

Therefore $|x+1||x-2| < (x-1)(-x+2) \leq |(x+1)(-x+2)| = |x+1||x-2|$. Contradiction arises.

**Remark.** We may simply quote the Triangle Inequality in the argument:

Suppose it were true that there existed some $x \in \mathbb{R}$ such that $|x+4| > 2|x+1| + |x-2|$.
By Triangle Inequality, we have $|x+4| = |2(x+1) + (-x+2)| \leq |2(x+1)| + |-x+2| = 2|x+1| + |x-2|$.
Then $|x+4| \leq 2|x+1| + |x-2| < |x+4|$. Contradiction arises.

(d) Suppose it were true that there existed some $z, w \in \mathbb{C}$ such that $w \neq 2z$ and $\dfrac{2|z-2w-3-6i| + 3|w+2+4i|}{|2z-w|} < 1$.

By the Triangle Inequality, we have

$$\begin{aligned} 2|z-2w-3-6i| + 3|w+2+4i| &= |2z-4w-6-12i| + |3w+6+12i| \\ &\geq |(2z-4w-6-12i) + (3w+6+12i)| = |2z-w|. \text{—— } (\dagger) \end{aligned}$$

Since $w \neq 2z$, we would have $2z - w \neq 0$. Then $|2z-w| > 0$.

Then by $(\dagger)$, we would have $\dfrac{2|z-2w-3-6i| + 3|w+2+4i|}{|2z-w|} \geq 1$. Contradiction arises.

*Alternative argument.*
We prove the statement

'*For any* $z, w \in \mathbb{C}$, *if* $w \neq 2z$ *then* $\dfrac{2|z-2w-3-6i| + 3|w+2+4i|}{|2z-w|} \geq 1$'.

Pick any $z, w \in \mathbb{C}$. Suppose $w \neq 2z$.
By the Triangle Inequality, we have

$$\begin{aligned} 2|z-2w-3-6i| + 3|w+2+4i| &= |2z-4w-6-12i| + |3w+6+12i| \\ &\geq |(2z-4w-6-12i) + (3w+6+12i)| = |2z-w|. \text{—— } (\dagger) \end{aligned}$$

Since $w \neq 2z$, we have $2z - w \neq 0$. Then $|2z-w| > 0$.

Then by $(\dagger)$, we have $\dfrac{2|z-2w-3-6i| + 3|w+2+4i|}{|2z-w|} \geq 1$.

17. **Solution.**

(a) *Method (A).*
The negation of $(\star)$ reads:

$\sim(\star)$: For any positive real numbers $x, y$, the inequality $(x+y)^2 > x^2 + y^2$ holds.

We verify $\sim(\star)$:

- Pick any positive real numbers $x, y$.
Note that $(x+y)^2 - x^2 - y^2 = 2xy$. —— $(\star)$
Since $x > 0$ and $y > 0$, we have $2xy > 0$.
Then, by $(\star)$, we have $(x+y)^2 - x^2 - y^2 > 0$.
Therefore $(x+y)^2 > x^2 + y^2$.

*Method (B).*

[We dis-prove the statement $(\star)$ by obtaining a contradiction from it.]

Suppose there existed some positive real numbers $x, y$ such that $(x+y)^2 \leq x^2 + y^2$.

Then, for the same $x, y$, we would have $xy = \dfrac{1}{2}[(x+y)^2 - x^2 - y^2] \leq 0$.

Since $x > 0$ and $y > 0$, we have $xy > 0$.
Then $0 < xy \leq 0$. Contradiction arises.

(b) [We disprove the statement ($\star\star$) by obtaining a contradiction from it.]

Suppose it were true that there existed some positive real numbers $u, v$ such that $\sqrt{u} + \sqrt{v} \leq \sqrt{u+v}$.

We claim that ($\star$) would hold:

- Define $x = \sqrt{u}$, $y = \sqrt{v}$. By definition, $x, y$ would be positive real numbers. Then $x + y$ would be a positive real number also.
  Therefore we would have $0 < x + y = \sqrt{u} + \sqrt{v} \leq \sqrt{u+v}$.
  Since $\sqrt{u+v} > 0$ and $u = x^2$ and $v = y^2$, we would have $(\sqrt{u+v})^2 = u + v = x^2 + y^2$.
  Then $(x+y)^2 \leq (\sqrt{u+v})^2 = u + v = x^2 + y^2$.
  Therefore ($\star$) would hold.

However, ($\star$) is a false statement. Contradiction arises.

18. **Solution.**

Suppose there existed some $k \in \mathbb{N}\backslash\{0,1\}$ such that for any positive integer $n$, the number $k^{1/n}$ was an integer.

Define the set $S$ by

$$S = \left\{ x \in \mathbb{N}\backslash\{0,1\} : \begin{array}{l} \text{There exists some } n \in \mathbb{N}\backslash\{0\} \\ \text{such that } x = k^{1/n} \end{array} \right\}.$$

By definition, $S$ would be a subset of $\mathbb{N}$.

Note that $k = k^{1/1}$, and $k \neq 0$ and $k \neq 1$. Then, by definition, $k \in S$. Therefore $S$ is a non-empty subset of $\mathbb{N}$.

Then, by the Well-ordering Principle for Integers, $S$ has a least element, say, $u$.

By definition, $u \geq 2$.

Also, by definition, there exists some $n_0 \in \mathbb{N}\backslash\{0\}$ such that $u = k^{1/n_0}$.

Note that $u \neq 0$; otherwise we would have $k = 0$.

Now define $v = k^{1/(2n_0)}$.

Note that $2n_0 \in \mathbb{N}\backslash\{0\}$. By definition, $v \in S$. Moreover, $u = v^2$.

Since $u \neq 0$, we would have $v \neq 0$; otherwise, $u = v^2 = 0$.

Also, since $u \neq 1$, we would have $v \neq 1$; otherwise $u = v^2 = 1$.

Then $v \geq 2$.

Then, since $u, v \in \mathbb{N}$, we would have $u = v \cdot v \geq 2v > v$.

But $u$ was a least element of $S$. Contradiction arises.