MATH1050BC/1058 Assignment 4

Advice.

1. The questions in this assignment are about set language, quantifiers, and formats of arguments of dis-proofs. Do familiarize yourself with the corresponding material available in the course homepage before trying the questions.

2. (a) Questions (1)-(3), (12)-(18) are mainly about quantifiers, and about dis-proofs.

   (b) Questions (4)-(11), (15) are mainly about set language.

3. Questions which require more thought and/or work and/or tricks and/or organization and/or ... than the 'unlabelled' questions are labelled by $\diamondsuit, \clubsuit, \heartsuit, \spadesuit$ in ascending order of overall difficulty level.

Instructions.

1. Any work submitted by you must be written on **A4-size sheets** and must be appropriately binded.

   Your name and student ID, as in your student card, and the **code** of the section to which you are registered must be written at the upper right corner of the first page of your submission.

2. (a) **Mandatory work, for assessment purpose.**

   You are **required** to submit work on Questions (1), (2), (3c), (3d), (4), (5a), (6), (9), (12) for course assessment purpose.

   (b) **Optional proof-writing exercise.**

   You may also **opt** to submit, **on exactly one sheet, separate from your submission on mandatory work**, your work on Questions (7a), (10a), (13a), (16b). It will be read and commented, but not counted for course assessment.

   (c) **Other optional work.**

   You may choose to submit work on other questions in this assignment not mentioned above, alongside the mandatory work, but there is no guarantee that it will be read.

3. (a) You must adhere to the notations which have been introduced in the course. Things which have not been formally defined in the course are not allowed in your work.

   (b) When you are giving a proof for something, you must make it clear what you are assuming (and indicate your assumptions with the use of appropriate words at appropriate places). You are also expected to indicate the 'flow of logic' in the argument with correct and appropriate use of words/symbols.

   (c) When you are dis-proving a statement, you had better first figure out what the negation of the statement under consideration is, and you are responsible for making it clear what you are trying to verify.

   (d) Words of the likes of '*trivial*', '*obvious*', '*clear(-ly)*' are not allowed to appear in your work.

*   *   *

1. For each statement below, write down its negation in such a way that the negation symbol can only appear immediately to the left of the symbols standing for the atomic predicates such as $P(x)$, $Q(y)$, $R(z)$, $S(x,y)$, $T(x,y,z)$.

   (a) $(\forall a)(\exists b)(\forall c)[P(c) \longrightarrow [Q(a) \wedge R(b)]]$.

   (b) $(\exists a)(\forall b)(\exists c)[P(a,c) \wedge [Q(a,b,c) \longrightarrow R(a,b,c)]]$.

   (c) $(\forall a)(\forall b)(\exists c)(\exists d)[[P(a,c) \vee Q(b,d)] \wedge R(c,d)]$.

   (d) $(\forall a)(\forall b)[P(a,b) \longrightarrow [(\exists c)(\forall d)[Q(a,b,c,d) \wedge R(a,b,c,d)]]]$.

   (e) $(\forall a)(\exists b)[[(\forall c)(\exists d)[P(a,b,c,d) \longrightarrow Q(a,b,c,d)]] \wedge [(\forall e)(\exists f)[(\sim S(a,b,e,f)) \longrightarrow T(a,b,e,f)]]]$.

   (f) $(\forall a)[[(\exists b)(\forall c)P(b,c) \longrightarrow Q(a,b,c)] \longrightarrow [(\forall d)(\exists e)[R(a,d,e) \longrightarrow [S(a,d,e) \wedge T(a,d,e)]]]]$.

2. For each statement below, write down its negation in such a way that the word '*not*' does not explicitly appear. (Do not worry about the mathematical content.)

   (a) *There exists some $\zeta \in \mathbb{C}$ such that for any $\eta \in \mathbb{C}$, ($|\zeta| < |\eta|$ and $|\zeta + \eta| \leq |\zeta - \eta|$).*

   (b) *For any $x \in \mathbb{R}$, there exist some $s, t \in \mathbb{Q}$ such that for any $n \in \mathbb{Z}$, if $s < n < t$ then $|x - n| > |t - s|$.*

   (c) *For any $p \in \mathbb{R}$, there exists some $q \in \mathbb{R}$, $n \in \mathbb{N}$ such that for any $s, t \in \mathbb{R}$, if $|s - t| < |q|$ then $|s^n - t^n| < |p|$.*

   (d) *For any $s, t \in \mathbb{Q}$, there exist some $p, q \in \mathbb{R}$ such that for any $n \in \mathbb{Z}$, (if $|s - t| \leq |q|$ then ($t^n \leq |p|$ and $s^n \leq |p|$)).*

   (e) *There exists some $n \in \mathbb{N}$ such that (for any $\varepsilon \in (0, +\infty)$, there exists some $\delta \in (0, +\infty)$ such that (for any $u, v \in \mathbb{C}$, if $|u - v| < \delta$ then $|u^n - v^n| < \varepsilon$)).*

(f) *For any $p, q \in \mathbb{Z}$, there exist some $s, t \in \mathbb{Z}$ such that for any $m, n \in \mathbb{N}$, if $|p+q| \geq s^m$ then ($|p^n - q| < t$ or $|p - q^n| < t$).*

(g) *For any $z \in \mathbb{C}$, there exists some $r \in \mathbb{R}$ such that for any $w \in \mathbb{C}$, (if $|z-w| \leq r$ then ($z \notin \mathbb{R}$ and $|w| > r$)).*

(h) *For any $z, w \in \mathbb{C}$, if $|z-w| \geq |z+w|$ then (there exists some $s \in \mathbb{R}$ such that (for any $t \in \mathbb{R}$, ($|z-s-t| > |w|$ or $|z| < 1$))).*

(i) *For any $\zeta, \alpha, \beta \in \mathbb{C}$, if (there exist some $s, t \in \mathbb{R}$ such that $\zeta = s\alpha + t\beta$) then (there exist some $p, q \in \mathbb{R}$ such that $\zeta = p\bar{\alpha} + q\bar{\beta}$).*

(j)$^\diamondsuit$ *Let $\zeta, \alpha, \beta, \gamma \in \mathbb{C}$. Suppose $\alpha\zeta^2 + \beta\zeta + \gamma = 0$. Then for any $s, t \in \mathbb{R}$, there exists some $r \in \mathbb{R}$ such that $\zeta = r\alpha + s\beta + t\gamma$.*

(k)$^\diamondsuit$ *Let $\zeta \in \mathbb{C}$. Suppose (for any $\alpha \in \mathbb{C}$, there exist some $\beta, \gamma \in \mathbb{C}$ such that $\alpha\zeta^2 + \beta\zeta + \gamma = 0$). Then for any $\rho, \sigma \in \mathbb{C}$, for any $s, t \in \mathbb{R}$, there exist some $\tau \in \mathbb{C}$, $r \in \mathbb{R}$ such that $\zeta = r\rho + s\sigma + t\tau$.*

(l)$^\clubsuit$ *Let $\zeta \in \mathbb{C}$. Suppose $|\zeta| > 1$. Further suppose (for any $\beta, \gamma \in \mathbb{C}$, if $\beta\gamma \neq 0$ then there exists some $\alpha \in \mathbb{C}$ such that $\alpha\zeta^2 + \beta\zeta + \gamma = 0$). Then for any $\rho, \sigma \in \mathbb{C}$, for any $s, t \in \mathbb{R}$, if $st \neq 0$ and $\rho + \sigma \neq 0$ then there exist some $\tau \in \mathbb{C}$, $r \in \mathbb{R}$ such that $\zeta = r\rho + s\sigma + t\tau$.*

3. *This is a review question on the framework of proofs of statements starting with* 'there exists'*.*

   Prove each of the 'existence statements' below. (The proofs are easy: conceive the candidates and verify the candidacy. Do not think too hard.)

   (a) *There exists some $n \in \mathbb{N}$ such that $n, n+2, n+4$ are prime numbers.*

   (b) *There exists some $x \in \mathbb{R}$ such that $x^2 - 2 = 0$.*

   (c) *There exists some $z \in \mathbb{C}$ such that $z^4 = -1$.*

   (d) *There exists some $x \in \mathbb{Q}$ such that $(\log_2(-2x))^2 = -\log_2(4x^2)$.*

4. *You are not required to justify your answers in this question.*

   Let

$$A = \Big\{\{0,1\}, \{1\}, \{1\}, \{1,2,3\}, \{3,4\}\Big\}, \quad B = \Big\{\{0,1,1,1\}, \{1,2,3\}, \{\{3\},\{4\}\}\Big\},$$
$$C = \Big\{\{\emptyset\}, \mathbb{N}\Big\}, \quad D = \Big\{\emptyset, \mathbb{N}, \mathbb{Z}, \{\mathbb{Q}\}\Big\}, \quad E = \Big\{\emptyset, \{\mathbb{N}\}, \{\mathbb{Z}\}, \{\mathbb{Q}\}\Big\}, \quad F = \Big\{\emptyset, \{\mathbb{N}, \mathbb{Z}\}, \{\mathbb{Z}, \mathbb{Q}\}\Big\}.$$

   For each set below, decide whether it is the empty set or not, and list its elements (if any), each exactly once.

   (a) i. $A \cap B$. ii. $A \cup B$. iii. $A \backslash B$. iv. $B \backslash A$. v. $A \triangle B$. vi. $(A \cap B) \times (A \backslash B)$. vii. $\mathfrak{P}(A \backslash B)$. viii. $\mathfrak{P}((\mathfrak{P}(B \backslash A) \backslash \{\emptyset\}))$.

   (b) i. $C \cup D$ ii. $D \cap E$ iii. $F \backslash E$ iv. $(C \cup E) \backslash (D \cup F)$ v. $\mathfrak{P}(C)$ vi. $\mathfrak{P}(D) \cap \mathfrak{P}(E)$ vii. $C \times F$ viii. $(D \cap E)^2$

   - If the set concerned is the empty set, write your answer as: '*The set* <u>*blah-blah-blah*</u> *is the empty set.*' (Example: *The set $A \backslash A$ is the empty set.*)
   - If the set concerned is not the empty set, write: '*There are exactly* <u>*bleh-bleh-bleh*</u> *elements in the set* <u>*blah-blah-blah*</u>*. They are* <u>*blih-blih-blih*</u>*.*' (Example: *There are exactly four elements in the set A. They are $\{0,1\}, \{1\}, \{1,2,3\}, \{3,4\}$.*)

5. (a) Let $A = \left\{ x \in \mathbb{N} \backslash \{0,1,2,3,4,5\} : \begin{array}{l} \text{There exists some } k \in \mathbb{N} \\ \text{such that } x = k^3/5^2. \end{array} \right\}$, $B = \left\{ x \in \mathbb{N} \backslash \{0,1,2,3,4,5\} : \begin{array}{l} \text{For any } k \in \mathbb{N}, \\ x = k^3/5^2. \end{array} \right\}$.

   For each set above, decide whether the set concerned is the empty set or not. Justify your answer.

   (b)$^\diamondsuit$ Let $C = \left\{ x \in \mathbb{N} \backslash \{0,1,2,3,4,5\} : \begin{array}{l} \text{There exists some } k \in \mathbb{N} \\ \text{such that } 5^2 x^2 = \left(\sqrt[3]{k}\right)^4. \end{array} \right\}$, $D = \left\{ x \in \mathbb{N} \backslash \{0,1,2,3,4,5\} : \begin{array}{l} \text{For any } k \in \mathbb{N}, \\ 5^2 x^2 = \left(\sqrt[3]{k}\right)^4. \end{array} \right\}$.

   For each set above, decide whether the set concerned is the empty set or not. Justify your answer.

6. (a) i. Explain the phrase '*subset of a set*' by stating the appropriate definition.

   ii. Complete the passage below in such a way that it provides an explanation for the phrase '*being not a subset of a set*' in terms of the notion of belonging:

   Suppose ___(I)___ *are sets. Then C is not a subset of D iff* ______(II)______.

(b) Let $A = \left\{ x \,\middle|\, \begin{array}{l} \text{There exists some } m \in \mathbb{Z} \\ \text{such that } x = 16m^6 \end{array} \right\}$, $B = \left\{ x \,\middle|\, \begin{array}{l} \text{There exists some } m \in \mathbb{Z} \\ \text{such that } x = 2m^3 \end{array} \right\}$.

Denote by (I), (J) the respective statements below: (I) $A \subset B$. (J) $B \not\subset A$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs for the statements $(I), (J)$ in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

(*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. Here we prove the statement (I):—

> [We intend to deduce the statement '_____(I)_____']
>
> Pick any object x. Suppose ____(II)____.
> By the definition of A, ____(III)____ $m \in$ __(IV)__ such that ____(V)____.
>
> [We now ask whether it is true that $x \in B$.]
>
> Define $n =$ __(VI)__. Note that ____(VII)____ and $n = 2m^2$. Then $n \in$ __(VIII)__.
> By definition, $x =$ _____(IX)_____.
> Hence, by the definition of B, ____(X)____.
> It follows that $A \subset B$.

ii. Here we prove the statement (J):—

> [We intend to deduce the statement '_____(I)_____']
>
> Take $x_0 = 2$.
> We verify that $x_0 \in B$:
>
> - Note that ___(II)___ and __(III)__ $\in \mathbb{Z}$. Then, by the definition of B, ___(IV)___.
>
> We verify that $x_0 \notin A$, by applying the method of proof-by-contradiction:
>
> - ______(V)______. Then _____(VI)_____ such that $x_0 = 16m^6$.
>   Now $2 = x_0 = 16m^6$. Then ___(VII)___ $= 8m^6 =$ ___(VIII)___.
>   Note that $4, m \in \mathbb{Z}$. Then ___(IX)___.
>   Therefore by the definition of divisibility, 1 would be ___(X)___. Contradiction arises.
>   It follows that, in the first place, $x_0 \notin A$.
>
> Hence $B \not\subset A$.

(c) Let $A = \left\{ \zeta \in \mathbb{C} : |\zeta| \leq 2 \right\}$, $B = \left\{ \zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| \leq 2 \text{ and } |\mathsf{Im}(\zeta)| \leq 2 \right\}$.

Denote by (K), (L) the respective statements below: (K) $A \subset B$. (L) $B \not\subset A$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* proofs for the statements $(K), (L)$ in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* proof for each respective statement.

i. Here we prove the statement (K):—

> [We intend to deduce the statement '_____(I)_____']
>
> Pick any $\zeta \in \mathbb{C}$. ____(II)____. By the definition of A, ___(III)___. —— $(\star)$
>
> [We now ask whether it is true that $\zeta \in B$.]
>
> We verify $|\mathsf{Re}(\zeta)| \leq 2$:—
>
> - We have $|\mathsf{Re}(\zeta)|^2 = (\mathsf{Re}(\zeta))^2$ __(IV)__ $(\mathsf{Re}(\zeta))^2 +$ __(V)__ $= |\zeta|^2$.—— $(\star\star)$
>   Note that $|\mathsf{Re}(\zeta)|$ and $|\zeta|$ are non-negative.
>   Then by $(\star\star)$ the inequality ___(VI)___ holds. Now by $(\star)$, the inequality __(VII)__ holds.
>
> We also verify $|\mathsf{Im}(\zeta)| \leq 2$: ______(VII)______
> Therefore $|\mathsf{Re}(\zeta)| \leq 2$ __(IX)__ $|\mathsf{Im}(\zeta)| \leq 2$. Hence, by the definition of B, ____(X)____.
> It follows that $A \subset B$.

ii. Here we prove the statement (L):—

> [We intend to deduce the statement '______(I)______']
>
> ____(II)____
>
> We verify that $\zeta_0 \in B$:
>
> - We have __(III)__ $= 2$. Then $|\mathsf{Re}(\zeta_0)| \leq 2$.
>   We have $\mathsf{Im}(\zeta_0) = 2$. Then ___(IV)___ .
>   Therefore $|\mathsf{Re}(\zeta_0)| \leq 2$ and $|\mathsf{Im}(\zeta_0)| \leq 2$. Hence, by the definition of B, we have __(V)__ .
>
> We verify that $\zeta_0 \notin A$:
>
> - We note that __(VI)__ $= (\mathsf{Re}(\zeta_0))^2 + (\mathsf{Im}(\zeta_0))^2 =$ ____(VII)____ $>$ __(VIII)__ .
>   Then $|\zeta_0| >$ __(IX)__ . Therefore, by the definition of A, ___(X)___ .
>
> Hence $B \not\subset A$.

7. *In this question, make good use of the* **Triangle Inequality on the Argand plane**. *Try not to 'break' a complex number into its real and imaginary parts.*

   (a) Let $A = \left\{\zeta \in \mathbb{C} : |\zeta - i| < 1\right\}$, $B = \left\{\zeta \in \mathbb{C} : |\zeta + i| < 3\right\}$.

   i. Is it true that $A \subset B$? Justify your answer.
   ii. Is it true that $B \subset A$? Justify your answer.

   (b) Let $D = \left\{\zeta \in \mathbb{C} : |\zeta| \leq 5\right\}$, $E = \left\{\zeta \in \mathbb{C} : |\zeta - 4| + |\zeta + 4| \leq 10\right\}$, $F = \left\{\zeta \in \mathbb{C} : |\zeta| \leq 3\right\}$.

   i.$^\diamondsuit$ Is it true that $D \subset E$? Justify your answer.
   ii.$^\diamondsuit$ Is it true that $E \subset D$? Justify your answer.
   iii.$^\diamondsuit$ Is it true that $E \subset F$? Justify your answer.
   iv.$^\clubsuit$ Is it true that $F \subset E$? Justify your answer.

8. (a) Let $A = \left\{x \,\middle|\, \begin{array}{l}\text{There exists some } n \in \mathbb{Z} \\ \text{such that } 3x = 8n + 1\end{array}\right\}$, $B = \left\{x \,\middle|\, \begin{array}{l}\text{There exists some } n \in \mathbb{Z} \\ \text{such that } 9x = 4n - 1\end{array}\right\}$.

   i. Is it true that $A \subset B$? Justify your answer.
   ii. Is it true that $B \subset A$? Justify your answer.

   (b)$^\diamondsuit$ Let $C = \left\{x \,\middle|\, \begin{array}{l}\text{There exist some } m, n \in \mathbb{Z} \\ \text{such that } x = 12m + 18n\end{array}\right\}$, $D = \left\{x \,\middle|\, \begin{array}{l}\text{There exist some } m, n \in \mathbb{Z} \\ \text{such that } x = 6m + 8n\end{array}\right\}$.

   i. Is it true that $C \subset D$? Justify your answer.
   ii. Is it true that $D \subset C$? Justify your answer.

9. (a) Explain the phrases below by stating the appropriate definitions.

   i. *set equality*
   ii. *empty set*
   iii. *intersection of two sets*
   iv. *union of two sets*
   v. *complement of one set in another set*
   vi. *power set of a set*

   (b) Denote by $(A), (B), (C), (D)$ the statements below:—

   (A) *Let A, B be sets. Suppose $A \cup B \subset B$. Then $A \backslash B = \emptyset$.*
   (B) *Suppose C, D are sets. Then $\mathfrak{P}(C) \cup \mathfrak{P}(D) \subset \mathfrak{P}(C \cup D)$.*
   (C) *Suppose A, B are sets. Then $A \cap B \subset A$. Moreover, $A \cap B = A$ iff $A \subset B$.*
   (D) *Let A, B, C be sets. Suppose $A \subset C$ and $B \subset C$. Then $A \subset B$ iff $C \backslash B \subset C \backslash A$.*

   For each statement:—

   - **either** (1) fill in the blanks (all labelled by capital-letter Roman numerals) in its *partially completed* proof in the corresponding block below, with appropriate words/symbols so as to obtain a *complete* proof for the statement,
   - **or** (2) write your own proof for the statement concerned, with direct reference to the respective definitions of subset relation, set equaltiy, and the various set operations.

   (*Note that the 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. This is a partially completed proof for the statement (A). (*Complete it, or write your own proof.*)

> Let A, B be sets. Suppose ___(I)___ . Further suppose ___(II)___ .
> Take some ___(III)___ . We would have ___(IV)___ $x_0 \notin B$ by definition of complement.
> In particular $x_0 \in A$. Then ___(V)___ or ___(VI)___ . Therefore ___(VII)___ by the definition of union.
> Since $x_0 \in A \cup B$ and $A \cup B \subset B$, we would have ___(VIII)___ by the definition of subset relation.
> Now we have $x_0 \in B$ ___(IX)___ $x_0 \notin B$. ___(X)___ .
> It follows that ___(XI)___ in the first place.

ii. This is a partially completed proof for the statement (B). (*Complete it, or write your own proof.*)

> ___(I)___
> Pick any object S. ___(II)___ . Then ___(III)___ , by the definition of union.
>
> - (Case 1). ___(IV)___ . Then ___(V)___ by the definition of power set.
>
>   We verify that $S \subset C \cup D$ (according to the definitions of subset relation and union):
>
>   * Pick any object x. Suppose $x \in S$. ___(VI)___ . Then ___(VII)___ .
>   Therefore by the definition of union, we have ___(VIII)___ .
>
>   Since ___(IX)___ , we have ___(X)___ by the definition of power set.
>
> - (Case 2). ___(XI)___ Modifying the argument in Case (1), we also deduce $S \in \mathfrak{P}(C \cup D)$.
>
> Hence, in any case, we have ___(XII)___
> It follows that $\mathfrak{P}(C) \cup \mathfrak{P}(D) \subset \mathfrak{P}(C \cup D)$.

iii. This is a partially completed proof for the statement (C). (*Complete it, or write your own proof.*)

> Suppose A, B are sets.
>
> [We want to verify '$A \cap B \subset A$'. According to definition, it is: '*For any object x,* ___(I)___ .']
>
> Pick any object x. ___(II)___ .
> Then, by the definition of intersection, ___(III)___ . Therefore ___(IV)___ in particular.
> It follows that $A \cap B \subset A$.
>
> - [We want to verify '*if $A \subset B$ then $A \cap B = A$*'.]
>   Suppose $A \subset B$.
>   [Under this assumption, we verify '$A \subset A \cap B$'. According to definition, it is: '___*(V)*___ .']
>   ___(VI)___
>   Since ___(VII)___ , we have $x \in B$ by the definition of subset relation.
>   Now we have ___(VIII)___ (simultaneously).
>   Then, by the definition of intersection, ___(IX)___ .
>   It follows that $A \subset A \cap B$.
>   Recall that ___(X)___ also. Then $A \cap B = A$.
>
> - [We want to verify '*if $A \cap B = A$ then $A \subset B$*'.]
>   Suppose ___(XI)___ .
>   [Under this assumption, we verify '$A \subset B$'. According to definition, it is: ' ___(XII)___ ']
>   Pick any object x. Suppose ___(XIII)___ .
>   Since ___(XIV)___ , we have $x \in A \cap B$. Then, ___(XV)___ . In particular, ___(XVI)___ .
>   It follows that $A \subset B$.
>
> Hence $A \cap B = A$ iff $A \subset B$.

iv. This is a partially completed proof for the statement (D). (*Complete it, or write your own proof.*)

Let A, B, C be sets. Suppose $A \subset C$ and $B \subset C$.

- [We want to deduce '*if* $A \subset B$ *then* $C \backslash B \subset C \backslash A$.']
  Suppose $A \subset B$.
  [Under this assumption, we verify '$C \backslash B \subset C \backslash A$'. It is: '*for any object* x, *if* $x \in C \backslash B$ *then* $x \in C \backslash A$.']
  Pick any object x. ______(I)______
  By the definition of ___(II)___, we have ______(III)______.
  We verify that $x \notin A$, with the method of proof-by contradiction:
  - ______(IV)______
    Then, ___(V)___ $A \subset B$, we would have ___(VI)___ by the definition of subset relation.
    Recall $x \notin B$. Then $x \in B$ ___(VII)___ $x \notin B$. Contradiction arises.

  Therefore ______(VIII)______.
  Hence, by the definition of complement, we have ___(IX)___.
  It follows that $C \backslash B \subset C \backslash A$.
- [We want to deduce '*if* $C \backslash B \subset C \backslash A$ then $A \subset B$.']
  Suppose $C \backslash B \subset C \backslash A$. [Under this assumption, we verify '$A \subset B$'.]
  ______(X)______ —— $(*)$
  We deduce that $x \in B$ with the method of proof-by-contradiction:
  - ______(XI)______
    Since $x \in A$ ______(XII)______, we have ___(XIII)___ by the definition of subset relation.
    Now $x \in C$ ___(XIV)___ (simultaneously). Then ___(XV)___ by the definition of complement.
    Then, since $x \in C \backslash B$ and $C \backslash B \subset C \backslash A$, we have ___(XVI)___ by the definition of subset relation.
    Then, by the definition of ___(XVII)___, we have ______(XVIII)______.
    In particular, ___(XIX)___.
    Recall that by assumption $(*)$, $x \in A$. Then ______(XX)______. Contradiction arises.

  Hence, in the first place, we have $x \in B$.
  It follows that $A \subset B$.

Hence $A \subset B$ iff $C \backslash B \subset C \backslash A$.

10. Prove the statements below, with reference to the definitions of set equality, subset relation, and the various set operations as appropriate.
    (a) *Let* A, B, C, D *be sets. Suppose* $A \subset C$ *and* $B \subset D$. *Then* $A \cup B \subset C \cup D$.
    (b) *Let* A, B, C *be sets. Suppose* $A \subset B$, $B \subset C$, *and* $C \subset A$. *Then* $A = B$.
    (c)$^\diamondsuit$ *Let* A, B *be sets. Suppose* $A \subset A \backslash B$. *Then* $A \cap B = \emptyset$.
    (d)$^\diamondsuit$ *Let* A, B *be sets. Suppose* $A \cap B = \emptyset$. *Then* $A \subset A \backslash B$.
    (e)$^\clubsuit$ *Let* A, B, C *be sets. Suppose* $A \subset C$ *and* $B \subset C$. *Then* $(C \backslash A) \backslash (C \backslash B) = B \backslash A$.
    (f)$^\diamondsuit$ *Let* A, B *be sets. Suppose* $\mathfrak{P}(B) \in \mathfrak{P}(A)$. *Then, for any subset* S *of* B, *the statement* '$S \in A$' *holds.*
11. We recall/introduce the definition for the notion of *proper subset*:—

    Suppose A, B *are sets. Then we say* A *is a* **proper subset** *of* B *if* ($A \subset B$ *and* $A \neq B$). *We write* $A \subsetneq B$.

    (a) Take for granted the equivalent formulation of this definition, in the form of the result ($\sharp$), (which can be proved immediately from the respective definitions):
    ($\sharp$) *Suppose* A, B *are sets. Then* A *is a proper subset of* B *iff* ($A \subset B$ *and* $B \not\subset A$).
    For each statement below, determine whether it is true of false. Justify your answer with an appropriate argument:—
      i. $\{1, 3, 5\}$ *is a proper subset of* $\{1, 3, 5, 7\}$.
      ii. $\{1, 3, 5, 9\}$ *is a proper subset of* $\{1, 3, 5, 7\}$.
      iii. $\{1, 1, 3, 5, 7\}$ *is a proper subset of* $\{1, 3, 5, 7, 7\}$.
    (b) Prove the statements below:—

i. *Suppose A, B are sets. Then $A \subsetneqq B$ iff ($A \subset B$ and $B \not\subset A$).*

ii. *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Further suppose $A \subsetneqq B$ or $B \subsetneqq C$. Then $A \subsetneqq C$.*

12. Denote by (R), (S), (T), (U), (V), (W) the respective statements below:—

(R) *Let $x, y, z \in \mathbb{Z}$. Suppose each of xy, xz is divisible by 4. Then xyz is divisible by 8.*

(S) *Let A, B, C be sets. Suppose $A \cap B \neq \emptyset$ and $A \cap B \subset C$. Then $A \subset C$ or $B \subset C$.*

(T) *Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$ and $|x^2 - 2x| < |y^2 - 2y|$. Then $x^2 \leq y^2$.*

(U) *Let $m, n \in \mathbb{N}\backslash\{0, 1, 2\}$ and $\zeta, \omega \in \mathbb{C}$. Suppose $m \neq n$ and $\zeta \neq \omega$. Further suppose that ζ is an m-th root of unity and ω is an n-th root of unity. Then $\zeta\omega$ is an $(m + n)$-th root of unity.*

(V) *There exist some $u \in \mathbb{R}\backslash\{-1, 0, 1\}$, $v \in \mathbb{R}$ such that $u^2 + v^2 \leq 2u^4$ and $u^6 + v^6 \leq 2v^4$.*

(W) *There exist some $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that ζ is both an 89-th root of unity and a 55-th root of unity.*

Fill in the blanks (all labelled by capital-letter Roman numerals) in the *partially completed* dis-proofs against the respective statements in the corresponding blocks below, with appropriate words/symbols so as to obtain a *complete* dis-proof against each respective statement.

(*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) We dis-prove the statement (R):—

[The negation of the statement (R) is given by:
($\sim R$) ______(I)______]

- We verify the negation of the statement (R) below:
  Take $x = 4$, ______(II)______ . Note that $x, y, z \in \mathbb{Z}$.
  Note that $xy =$ ___(III)___ , and $xz =$ ___(IV)___ .
  Note that ______(V)______ . Then xy is divisible by 4.
  By a similar argument, we also deduce that xz is divisible by 4.
  Note that $xyz =$ ___(VI)___ , which is not divisible by 8.
  Below is the justification of this claim:
  - Suppose it were true that ______(VII)______ .
    Then there would exist some $k \in \mathbb{Z}$ such that ______(VIII)______ .
    For the same k, we would have $k =$ ___(IX)___ , which is not an integer. Contradiction arises.

(b) We dis-prove the statement (S):—

[The negation of the statement (S) is given by:
($\sim S$) ______(I)______]

We verify the negation of the statement (S) below:

- Take $A = \{1, 3\}$, $B = \{2, 3\}$ and ___(II)___ .
  We have $A \cap B = \{3\}$. Then $A \cap B \neq$ ___(III)___ .
  Moreover $A \cap B = C$. Then ___(IV)___ .
  We verify $A \not\subset C$ and $B \not\subset C$:
  - We have $1 \in A$ ___(V)___ . Then ___(VI)___ .
    We have ______(VII)______ . Then ______(VIII)______ .
    Hence $A \not\subset C$ and $B \not\subset C$ (simultaneously).

(c) We dis-prove the statement (T):—

[The negation of the statement (T) is given by:
$(\sim T)$ ______________(I)______________]

We verify the negation of the statement (T) below:

- Take $x = 2$, ____(II)____ . Note that $x, y \in \mathbb{R}$, and _____(III)_____ .
  Note that $|x^2 - 2x| =$ ___(IV)___ and _____(V)_____ . Then ___(VI)___ $<$ ___(VII)___ .
  We have ____(VIII)____ and $y^2 = 1$. Then ____(IX)____ .

(d) We dis-prove the statement (U):—

[The negation of the statement (U) is given by:
$(\sim U)$ ______________(I)______________]

We verify the negation of the statement (U) below:

- ___(II)___ $m = 4$, $n = 8$, $\zeta = i$ and _____(III)_____ .
  We have $m, n \in \mathbb{N} \backslash \{0, 1, 2\}$ and $\zeta, \omega \in \mathbb{C}$. Also, _____(IV)_____ .
  Note that $\zeta^m = i^4 = 1$. Then _____(V)_____ .
  Note that ________(VI)________ . Then ω is an n-th root of unity.
  Now note that $m + n =$ ___(VII)___ and $\zeta\omega = \cos\left(\dfrac{3\pi}{4}\right) + i \sin\left(\dfrac{3\pi}{4}\right)$.
  We have ________(VIII)________ . Then $(\zeta\omega)^{m+n}$ ___(IX)___ 1.
  Therefore ______(X)______ .

(e) We dis-prove the statement (V):—

[We dis-prove the statement (V) by obtaining a contradiction from it.]

___(I)___ there existed some ____(II)____ , $v \in \mathbb{R}$ such that $u^2 + v^2 \leq 2u^4$ and _____(III)_____ .
For the same u, v, we would have $u^2 + v^2 - 2u^4 \leq 0$ and $u^6 + v^6 - 2v^4 \leq 0$.
Then $u^2(u^2 - 1)^2 + v^2(v^2 - 1)^2 =$ __________(IV)__________ .
Since u, v are real, $u^2(u^2 - 1)^2 \geq 0$ and _____(V)_____ ≥ 0.
Then $u^2(u^2 - 1)^2 = 0$ and _____(VI)_____ respectively.
In particular, _____(VII)_____ . Then $u = 0$ or $u = -1$ or $u = 1$. But _____(VIII)_____ .
Contradiction arises.

(f) We dis-prove the statement (W):—

[We dis-prove the statement (W) by obtaining a contradiction from it.]

__________(I)__________

For the same ζ, we would have $\zeta^{55} =$ ___(II)___ and ____(III)____ by the definition of root of unity.
(Note that $\zeta \neq 0$.) Then we would have $\zeta^{34} = \zeta^{89-55} = \zeta^{89}/\zeta^{55} = 1$.
Repeating the above argument, we would have:

__________(IV)__________

Recall that by assumption, $\zeta \in$ ___(V)___ . Then $\zeta \neq 1$.
Now $\zeta = 1$ ___(VI)___ $\zeta \neq 1$.
Contradiction arises.

13. Dis-prove each of the statements below. (It may help if you first find what the negation of the statement is.)

(a) *Let $x, y, z \in \mathbb{N}$. Suppose $x + y$, $y + z$ are divisible by 3. Then $x + z$ is divisible by 3.*

(b) *Let $x, y, z \in \mathbb{N}$. Suppose $x - y > 0$ and $x - z > 0$ and $x - z$, $y - z$ are divisible by 5. Then $x + y + z$ is not divisible by 5.*

(c) *Suppose $x, y \in \mathbb{N}$. Then $\sqrt{x^2 + y^2} \in \mathbb{N}$.*

(d)$^\diamond$ *For any $s, t \in \mathbb{R}$, if both of $s + t, st$ are rational, then at least one of s, t is rational.*

(e)$^\diamond$ *For any $a, b, c \in \mathbb{N}$, if ab is divisible by c and $c < a$ and $c < b$, then at least one of a, b is divisible by c.*

(f) *Let n be a positive integer, and ζ be a complex number. Suppose ζ is an n^2-th root of unity. Then ζ^2 is an n-th root of unity.*

(g) *Let n be a positive integer, and ζ be a complex number. Suppose ζ^n is an n-th root of unity. Then ζ is a $(2n)$-th root of unity.*

14. Dis-prove each of the statements below by giving an appropriate argument. (It may help if you draw Venn diagrams to investigate the respective statements first.)

(a) *Suppose A, B, C are sets. Then $A \backslash (C \backslash B) \subset A \cap B$.*

(b) *Suppose A, B, C are non-empty sets. Then $B \backslash A \subset (C \backslash A) \backslash (C \backslash B)$.*

(c) *Suppose A, B, C are non-empty sets. Then $A \cup (B \cap C) \subset (A \cup B) \cap C$.*

(d)$^\diamond$ *Suppose A, B, C are non-empty sets. Then $B \cap C \subset [A \backslash (B \backslash C)] \cup [B \backslash (C \backslash A)]$.*

(e)$^\diamond$ *Let A, B, C be sets. Suppose $A \cap B \subset C$. Then $C \subset (A \cap C) \cup (B \cap C)$.*

(f)$^\diamond$ *Let A, B, C be sets. Suppose $A \backslash B$, $A \backslash C$ are non-empty. Then $A \backslash (B \cap C) \subset (A \backslash B) \cap (A \backslash C)$.*

(g)$^\diamond$ *Let A, B, C, D be non-empty sets. Suppose $A \subset C$ and $B \subset D$. Further suppose $C \cap D \neq \emptyset$. Then $A \cup B \subset C \cap D$.*

15. Dis-prove the statements below:—

(a) *There exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.*

(b) *There exist some $x, y \in \mathbb{R} \backslash \{0\}$ such that $(x + y)^2 = x^2 + y^2$.*

(c)$^\diamond$ *There exists some $r \in \mathbb{R}$ such that $r < r^5 \leq r^3$.*

(d)$^\diamond$ *There exist some $\zeta \in \mathbb{C} \backslash \{1\}$, $n \in \mathbb{N} \backslash \{0, 1\}$ such that ζ is an $(n + 1)$-th root of unity and ζ is an $(n^2 + n + 1)$-th root of unity.*

(e)$^\diamond$ *There exists some $s \in \mathbb{Q}$ such that (for any $t \in \mathbb{Q}$, $s = 2t + 1$).*

(f)$^\diamond$ *There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).*

(g)$^\diamond$ *There exist some $a \in \mathbb{R}$, $n \in \mathbb{N} \backslash \{0, 1, 2, 3\}$ such that $\dfrac{(1 + \sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} \leq \dfrac{a^2}{24}$.*

16. Dis-prove the statements below. (Various results known as the Triangle Inequality may be useful.)

(a) *There exists some $x \in \mathbb{R}$ such that $|x + 1| > |x| + 1$.*

(b)$^\diamond$ *There exists some $z \in \mathbb{C}$ such that $|z + 3 - 4i| > |z| + 5$.*

(c)$^\diamond$ *There exists some $x \in \mathbb{R}$ such that $|x + 4| > 2|x + 1| + |x - 2|$.*

(d)$^\diamond$ *There exists some $z, w \in \mathbb{C}$ such that $w \neq 2z$ and $\dfrac{2|z - 2w - 3 - 6i| + 3|w + 2 + 4i|}{|2z - w|} < 1$.*

17. (a) Dis-prove the statement ($\star$):—

($\star$) *There exist some positive real numbers x, y such that $(x + y)^2 \leq x^2 + y^2$.*

(b)$^\diamond$ Hence, or otherwise, dis-prove the statement ($\star\star$):—

($\star\star$) *There exist some positive real numbers u, v such that $\sqrt{u} + \sqrt{v} \leq \sqrt{u + v}$.*

18.$^\clubsuit$ Dis-prove the statement ($\star$):—

($\star$) *There exists some $k \in \mathbb{N} \backslash \{0, 1\}$ such that for any positive integer n, the number $k^{1/n}$ is an integer.*

Remark. You will probably need the Well-ordering Principle for Integers.