MATH1050BC/1058 Assignment 3 (Answers and selected solution)

1. **Answer.**

   (a) i. True. ii. True. iii. True. iv. True. v. True. vi. True. vii. True. viii. True.

   (b) i. $2+3 \neq 6$ and $3+1 \neq 4$.
   ii. $2+3 > 6$ or $(3+1 < 4$ and $4+3 > 5)$.
   iii. $1+2 > 3$ and $3+4 > 2$.
   iv. $3+2 \geq 5$ and $2+1 > 3$.
   v. $2+6 \leq 5$ and $3+2 \neq 5$.
   vi. $2+4 \leq 3$ and $3+2 \neq 7$.
   vii. $2+3 \neq 6$ iff $1+4 = 7$.
   Alternative answer.
   $2+3 = 6$ iff $1+4 \neq 7$.
   viii. $[3+4 = 5$ and (if $3+4 = 5$ then $2+3 = 4)]$ and $2+3 \neq 4$.

2. **Answer.**

   (a) The statement $(P \longrightarrow R) \longrightarrow [(P \longrightarrow Q) \wedge (Q \longrightarrow R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$(P \to Q) \wedge (Q \to R)$	$(P \to R) \to [(P \to Q) \wedge (Q \to R)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	F
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

   Remark. Do you see what is wrong in the logic behind the word 'therefore' in the argument below?

   - If John visits his girlfriend then John puts aside his books. Therefore, if John visits his girlfriend then John plays football; furthermore, if John plays football then John puts aside his books.

   (b) The statement $(P \longrightarrow Q) \longrightarrow [(P \longrightarrow R) \vee (Q \longrightarrow R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

P	Q	R	$P \to Q$	$P \to R$	$Q \to R$	$(P \to R) \vee (Q \to R)$	$(P \to Q) \to [(P \to R) \vee (Q \to R)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

3. **Answer.**

   (a)

P	Q	$P \to Q$	$(P \to Q) \to Q$	$Q \to P$	$Q \to (Q \to P)$	$\sim P$	$(P \to Q) \to (\sim P)$	$Q \to (\sim P)$	$P \to [Q \to (\sim P)]$
T	T	T	T	T	T	F	F	F	F
T	F	F	T	T	T	F	T	T	T
F	T	T	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T	T	T

   (b) U is a contingent statement.

   (c) V is a contingent statement.

   (d) W is a contingent statement.

   (e) X is a contingent statement.

(f) $U \wedge V$ is logically equivalent to P.

(g) $W \longleftrightarrow X$ is a tautology.

(h) $U \longleftrightarrow W$ is logically equivalent to $\sim(P \longleftrightarrow Q)$.

4. **Answer.**

Let P, Q, R, S be statements, and denote the statements $(P \longrightarrow Q) \longrightarrow (R \longrightarrow S)$, $[P \longrightarrow (Q \longrightarrow R)] \longrightarrow S$ by U, V.

Suppose the truth value of U is F.

Then the truth value of $P \longrightarrow Q$ is T and the truth value of $R \longrightarrow S$ is F.

Since the truth value of $R \longrightarrow S$ is F, the truth value of R is T and the truth value of S is F.

Since the truth value of R is T, the truth value of $Q \longrightarrow R$ is T.

Since the truth value of $Q \longrightarrow R$ is T, the truth value of $P \longrightarrow (Q \longrightarrow R)$ is T.

Recall that the truth value of S is F. Then the truth value of V is F.

Since the truth values of U, V are both F, the truth value of $V \longrightarrow U$ is T.

5. **Answer.**

We verify that the truth value of $[(P \vee Q) \longrightarrow (Q \wedge R)] \longrightarrow (P \longrightarrow R)$ is T irrespective of the respective truth values of P, Q, R:

- Suppose it happened that the truth value $[(P \vee Q) \longrightarrow (Q \wedge R)] \longrightarrow (P \longrightarrow R)$ were F for some specific truth values of P, Q, R respectively.

  Then for the same truth values of P, Q, R, it would happen that the truth value of $(P \vee Q) \longrightarrow (Q \wedge R)$ was T and the truth value of $P \longrightarrow R$ was F.

  Since the truth value of $P \longrightarrow R$ was F, the truth value of P was T and the truth value of R was F.

  Since the truth value of R was F, the truth value of $Q \wedge R$ was F.

  Since the truth value of P was T, the truth value of $P \vee Q$ was T.

  Then the truth value of $(P \vee Q) \longrightarrow (Q \wedge R)$ would be F. This is impossible.

6. **Answer.**

(a) F

(b) Tautology.

Remark. This is the part of a truth table relevant to the explanation for this answer:

P	Q	$\sim P$	$\sim Q$	$P \longrightarrow (\sim Q)$	U	$U \longrightarrow (\sim P)$
F		T				T
T	T		F	F	F	T
T	F				F	T

(c) No.

Remark. When P, Q take the truth values F, T respectively, U takes the truth value T and $U \longrightarrow P$ takes the truth value F.

7. **Answer.**

(a) (I) there exist some integers m, n

(II) $n \neq 0$ and $m = nx$

(b) i. (I) There exist some integers s, t such that $t \neq 0$ and $s = t(xy)$.

(II) there exist some integers m, n such that

(III) such that $q \neq 0$ and $p = qy$

(IV) $nxqy = nq(xy)$

(V) $q \neq 0$

(VI) $nq \neq 0$

(VII) since m, n, p, q are integers

(VIII) nq

(IX) xy is rational

(X) Suppose x, y are rational

(XI) There exist some integers s, t such that $t \neq 0$ and $t = s(x + y)$.

(XII) there exist some integers m, n such that $n \neq 0$ and $m = nx$

(XIII) there exist some integers p, q such that $q \neq 0$ and $p = qy$

(XIV) Note that $mq + pn = nxq + qyn = nq(x + y)$. Since $n \neq 0$ and $q \neq 0$, we have $nq \neq 0$. Also, since m, n, p, q are integers, $mq + pn, nq$ are also integers.

(XV) $x + y$ is rational

ii. **Solution.**
Let x, y be real numbers. Suppose x, y are rational.

[We want to deduce that $x - y$ is rational.

This amounts to verifying this statement: '*there exist some integers s, t such that $t \neq 0$ and $s = t(x - y)$*'.]

By the definition of rational numbers, there exist some integers m, n such that $n \neq 0$ and $m = nx$.
Also, there exist some integers p, q such that $q \neq 0$ and $p = qy$.
Note that $mq - pn = nxq - qyn = nq(x - y)$.
Since $n \neq 0$ and $q \neq 0$, we have $nq \neq 0$.
Also, since $m, n, p, q \in$ are integers, $mq - pn, nq$ are also integers.
Hence, by definition of rational numbers, $x - y$ is rational.

iii. **Solution.**
Let x, y be real numbers. Suppose x, y are rational and $y \neq 0$.

[We want to deduce that $\dfrac{x}{y}$ is rational.

This amounts to verifying this statement: '*there exist some integers s, t such that $t \neq 0$ and $s = t\left(\dfrac{x}{y}\right)$*'.]

By the definition of rational numbers, there exist some integers m, n such that $n \neq 0$ and $m = nx$.
Also, there exist some integers p, q such that $q \neq 0$ and $p = qy$.
Since $y \neq 0$ and $q \neq 0$, we have $p \neq 0$.
Note that $mqy = nxp$.
Then $mq = np(\dfrac{x}{y})$.
Since $n \neq 0$ and $p \neq 0$, we have $np \neq 0$.
Also, since m, n, p, q are integers, mq, np are also integers.
Hence, by the definition of rational numbers, $\dfrac{x}{y}$ is rational.

8. (a) **Answer.**

i. Suppose x, y are integers. Then we say x is divisible by y if the statement (Div) holds:—

(Div) There exists some integer k such that $x = ky$.

ii. Let $P(x)$ be a predicate with variable x.
Suppose the statements (Ini), (Ind) hold:—

(Ini) $P(0)$ is true.

(Ind) For any natural number k, if $P(k)$ is true, then $P(k + 1)$ is true.

Then, for any natural number n, the statement $P(n)$ is true.

Acceptable answer.
Let S be a subset of $\mathbb{N}$.
Suppose the statements (Ini'), (Ind') below hold:—

(Ini') $0 \in \mathbb{N}$.

(Ind') For any $x \in \mathbb{N}$, if $x \in S$ then $x + 1 \in S$.

Then $S = \mathbb{N}$.

iii. Let S be a subset of $\mathbb{N}$. Suppose S is non-empty. Then S has a least element.
Acceptable answer.
Let S be a subset of $\mathbb{Z}$. Suppose S is non-empty. Further suppose that S is bounded below by some integer (not necessarily belonging to S). Then S has a least element.

(b) **Solution.**

i. For each natural number n, denote by $P(n)$ the proposition below:—
$n(n^2 + 2)$ *is divisible by* 3.

- [We verify $P(0)$.]
  Note that $0 \cdot (0^2 + 2) = 0 = 3 \cdot 0$, and 0 is an integer.
  Then, by the definition of divisibility, $0 \cdot (0^2 + 2)$ is divisible by 3.
  Hence $P(0)$ is true.

- [We verify the statement '*for any natural number k, if $P(k)$ is true then $P(k+1)$ is true*'.]
  Let k be a natural number. Suppose $P(k)$ is true. (Then $k(k^2+2)$ is divisible by 3.)
  [We verify that $P(k+1)$ is true.]
  By the definition of divisibility, there exists some integer q such that $k(k^2+2) = 3q$.
  Note that

$$(k+1)[(k+1)^2+2] = k^3+3k^2+5k+3 = k(k^2+2)+3(k^2+k+1) = \cdots = 3(q+k^2+k+1)$$

  Since k, q are integers, $q+k^2+k+1$ is an integer.
  Then, by the definition of divisibility, $(k+1)[(k+1)^2+2]$ is divisible by 3.
  Hence $P(k)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any natural number n.

ii. For each natural number n, denote by $P(n)$ the proposition below:—
$2^{4n+3}+3^{3n+1}$ *is divisible by* 11.

- [We verify $P(0)$.]
  Note that $2^{4\cdot 0+3}+3^{3\cdot 0+1} = 8+3 = 11 = 11\cdot 1$, and 1 is an integer.
  Then, by the definition of divisibility, $2^{4\cdot 0+3}+3^{3\cdot 0+1}$ is divisible by 11.
  Hence $P(0)$ is true.
- [We verify the statement '*for any natural number k, if $P(k)$ is true then $P(k+1)$ is true*'.]
  Let k be a natural number. Suppose $P(k)$ is true. (Then $2^{4k+3}+3^{3k+1}$ is divisible by 11.)
  [We verify that $P(k+1)$ is true.]
  By the definition of divisibility, there exists some integer q such that $2^{4k+3}+3^{3k+1} = 11q$.
  Note that

$$\begin{aligned} 2^{4(k+1)+3}+3^{3(k+1)+1} &= 16\cdot 2^{4k+3}+27\cdot 3^{3k+1} \\ &= 16\cdot(2^{4k+3}+3^{3k+1})+11\cdot 3^{3k+1} \\ &= 16\cdot 11q+11\cdot 3^{3k+1} = 11(16q+3^{3k+1}) \end{aligned}$$

  Since q, k are integers, $16q+3^{3k+1}$ is also an integer.
  Then, by the definition of divisibility, $2^{4(k+1)+3}+3^{3(k+1)+1}$ is divisible by 11.
  Hence $P(k)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any natural number n.

iii. For each natural number n, denote by $P(n)$ the proposition below:—
$7^n(3n+1)-1$ *is divisible by* 9.

- [We verify $P(0)$.]
  Note that $7^0(3\cdot 0+1)-1 = 0 = 9\cdot 0$, and 0 is an integer.
  Then, by the definition of divisibility, $7^0(3\cdot 0+1)-1$ is divisible by 9.
  Therefore $P(0)$ is true.
- [We verify the statement '*for any natural number k, if $P(k)$ is true then $P(k+1)$ is true*'.]
  Let k be a natural number. Suppose $P(k)$ is true. (Then $7^k(3k+1)-1$ is divisible by 9.)
  [We verify that $P(k+1)$ is true.]
  By the definition of divisibility, there exists some integer q such that $7^k(3k+1)-1 = 9q$.
  Note that

$$\begin{aligned} 7^{k+1}[3(k+1)+1]-1 &= 7\cdot 7^k[(3k+1)+3]-1 \\ &= 7\cdot 7^k(3k+1)+3\cdot 7^{k+1}-1 \\ &= 7\cdot[7^k(3k+1)-1]+3(7^{k+1}-1)+9 \\ &= 7\cdot 9q+3(7-1)\sum_{j=0}^{k}7^j+9 = 9\left(7q+2\sum_{j=0}^{k}7^j+1\right) \end{aligned}$$

  Since q, k are integers, $7q+2\displaystyle\sum_{j=0}^{k}7^j+1$ is also an integer.
  Then, by the definition of divisibility, $7^{k+1}[3(k+1)+1]-1$ is divisible by 9.
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any natural number n.

(c) **Solution.**

i. Let x, y be integers. Suppose that x is divisible by y, and y is divisible by x.
Since x is divisible by y, there exists some integer k such that $x = ky$.
Since y is divisible by x, there exists some integer ℓ such that $y = \ell x$.
We have $x = ky = k\ell x$.
Then $(k\ell - 1)x = 0$. —— $(\star)$
Note that $x = 0$ or $x \neq 0$.

- (Case 1.) Suppose $x = 0$. Then $y = \ell x = 0$. Therefore $|x| = 0 = |y|$.
- (Case 2.) Suppose $x \neq 0$. Then by $(\star)$, $k\ell - 1 = 0$.
  Therefore $k\ell = 1$. Then $k \neq 0$ and $\ell \neq 0$.
  Recall that k, ℓ are integers. Then $|k| \geq 1$ and $|\ell| \geq 1$.
  Now we have $|x| = |ky| = |k| \cdot |y| \geq 1 \cdot |y| = |y|$.
  We also have $|y| = |\ell x| = |\ell| \cdot |x| \geq 1 \cdot |x| = |x|$.
  Then $|x| = |y|$.

Hence in any case, $|x| = |y|$.

Remark. Be aware that the assumption 'x, y *are integers and* x *is divisible by* y *and* y *is divisible by* x' does not preclude the possibilities '$x = 0$', '$y = 0$'.

Be careful when you are tempted to write an expression like $\frac{y}{x}$, or 'cancelling' x from both sides of an equality in which the expressions contain the 'factor' x.

ii. Let x, y, z be integers. Suppose that x is divisible by y^2, and y is divisible by z^3.
By the definition of divisibility, since x is divisible by y^2, there exists some integer g such that $x = gy^2$.
Also, since y is divisible by z^3, there exists some integer h such that $y = hz^3$.
We have $x = gy^2 = g(hz^3)^2 = (gh^2)z^6$.
Since g, h are integers, gh^2 is also an integer.
Then, by the definition of divisibility, x is divisibly by z^6.

iii. Let x, n be integers. Suppose that x is divisible by n.
By definition of divisibility, there exists some integer k such that $x = kn$. —— $(\star)$

[We are going to verify that *for any integer* y, $(3x^2 + 4y)^5 + (3x^2 - 4y)^5$ *is divisible by* $6n^2$.]

Pick any integer y.

[For such an integer y, we are going to deduce (with the help of $(\star)$) that $(3x^2 + 4y)^5 + (3x^2 - 4y)^5$ *is divisible by* $6n^2$. This amounts to deducing (with the help of $(\star)$) that *there exists some integer* L *such that* $(3x^2 + 4y)^5 + (3x^2 - 4y)^5 = L \cdot 6n^2$.]

Note that

$$\begin{aligned}(3x^2 + 4y)^5 + (3x^2 - 4y)^5 &= [(3x^2 + 4y) + (3x^2 - 4y)]M \\ &= 6x^2M = 6(kn)^2M \quad \text{(by } (\star)\text{)} \\ &= 6n^2 \cdot k^2M,\end{aligned}$$

in which $M = (3x^2 + 4y)^4 - (3x^2 + 4y)^3(3x^2 - 4y) + (3x^2 + 4y)^2(3x^2 - 4y) - (3x^2 + 4y)(3x^2 + 4y) + (3x^2 - 4y)^4$.
Since x, y are integers, each of $3x^2 + 4y$, $3x^2 - 4y$ is an integer. Then M is an integer. Recall that k is an integer by $(\star)$. Then k^2M is also an integer.
Therefore, by the definition of divisibility, $(3x^2 + 4y)^5 + (3x^2 - 4y)^5$ is divisible by $6n^2$.

9. (a) **Solution.**

Let x, y, a, b, d be integers. Suppose that x is divisible to by d, and y is divisible by d.
By the definition of divisibility, since x is divisible by d, there exists some integer k such that $x = kd$.
Since y is divisible by d, there exists some integer ℓ such that $y = \ell d$.
Note that $ax + by = akd + b\ell d = (ak + b\ell)d$.
Since a, k, b, ℓ are integers, $ak + b\ell$ is also an integer.
Then by the definition of divisibility, $ax + by$ is divisible by d.

(b) i. **Answer.**

(I) d is an integer
(II) n greater than 1
(III) $a_1, a_2, \cdots, a_n, x_1, x_2, \cdots, x_n$
(IV) x_j is divisible by d for each $j = 1, 2, \cdots, n$

(V) $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is divisible by d

ii. **Solution.**
Suppose d is an integer.
For each integer n greater than 1, denote by $P(n)$ the proposition below:—
If $a_1, a_2, \cdots, a_n, x_1, x_2, \cdots, x_n$ are integers and x_j is divisible by d for each $j = 1, 2, \cdots, n$ then $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is divisible by d.
By $(*)$, $P(2)$ is true.

- [We now verify the statement '*for any integer k greater than 1, if $P(k)$ is true then $P(k+1)$ is true*'.]
  Let k be an integer greater than 1. Suppose $P(k)$ is true.
  [We deduce $P(k+1)$. It reads:
  '*If $a_1, a_2, \cdots, a_k, a_{k+1}, x_1, x_2, \cdots, x_k, x_{k+1}$ are integers and x_j is divisible by d for each $j = 1, 2, \cdots, k, k+1$ then $a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_{k+1}x_{k+1}$ is divisible by d.*']
  Pick any integers $a_1, a_2, \cdots, a_k, a_{k+1}, x_1, x_2, \cdots, x_k, x_{k+1}$.
  Suppose that x_j is divisible by d for each $j = 1, 2, \cdots, k, k+1$.
  Write $c = a_1x_1 + a_2x_2 + \cdots + a_kx_k$. By $P(k)$, c is divisible by d.
  Now note that $a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_{k+1}x_{k+1} = 1 \cdot c + a_{k+1}x_{k+1}$.——(†)
  By assumption, a_{k+1} is an integer, and x_{k+1} is divisible by d.
  Also note that 1 is an integer, and recall that c is divisible by d.
  Then by $(*)$, $1 \cdot c + a_{k+1}x_{k+1}$ is divisible by d.
  Therefore by (†), $a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_{k+1}x_{k+1}$ is divisible by d.
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any integer n greater than 1.

10. (a) **Answer.**

Let p be an integer. Suppose p is not amongst $0, 1, -1$. Then we say that p is a prime number if p is divisible by no integer other than $1, -1, p, -p$.

Acceptable answer.

Let p be an integer. Suppose p is not amongst $0, 1, -1$. Then we say that p is a prime number if the statement $(\star)$ holds:

$(\star)$ For any integer u, if p is divisible by u then $u = 1$ or $u = -1$ or $u = p$ or $u = -p$.

(b) **Solution.**

Pick any integer n greater than 3.

We verify that there is some integer v such that $n^2 + n - 6$ is divisible by v, and v is not amongst $1, -1, n^2 + n - 6, -(n^2 + n - 6)$.

Note that $n^2 + n - 6 = (n-2)(n+3)$.

Since n is an integer, each of $n - 2$, $n + 3$ is an integer.

By assumption, $n > 3$. Then $n - 2 > 1$ and $n + 3 > 1$.

Further note that $n^2 + n - 6 = (n-2)(n+3) > 1 \cdot (n+3) = n + 3 > 1 > 0$.

(Hence $n^2 + n - 6$ is not amongst $-1, 0, 1$.)

By the definition of divisibility, $n^2 + n - 6$ is divisible by $n + 3$.

Also, $n + 3$ is not amongst $-1, 1, n^2 + n - 6, -(n^2 + n - 6)$.

Therefore $n^2 + n - 6$ is not a prime number.

11. (a) **Answer.**

i. Suppose m, n, c are integers. Then we say that c is a common divisor of m, n if m is divisible by c and n is divisible by c.

ii. Let h, k, p be integers. Suppose p is a prime number. Further suppose hk is divisible by p. Then at least one of h, k is divisible by p.

iii. Let m, n be natural numbers. Suppose $n \neq 0$. Then there exists some unique natural numbers q, r such that $m = qn + r$ and $0 \leq r < n$.

(b) **Answer.**

i. (I) Let m be an integer greater than 1, and t, u, v be real numbers. Suppose that t is non-zero and rational, and u is irrational, and $v^m = tu$.

(II) v was rational

Alternative answer.
v was not irrational

(III) rational

(IV) $\dfrac{v^m}{t}$

(V) $\dfrac{v^m}{t}$

(VI) u would be rational

(VII) u is irrational

(VIII) Hence v is irrational in the first place.

ii. (I) Let a, p be integers. Suppose p is a prime number.

(II) If a^n is divisible by p, then a is divisible by p.

(III) Suppose a^2 is divisible by p.

(IV) Let k be an integer greater than 1. Suppose $Q(k)$ is true.

(V) Suppose a^{k+1} is divisible by p

(VI) By Euclid's Lemma, at least one of a, a^k is divisible by p

(VII) divisible by p

(VIII) by $Q(k)$, a is divisible by p

iii. (I) Suppose it were true that $\sqrt[5]{3}$ was rational.

Alternative answer.
Suppose $\sqrt[5]{3}$ were rational.

Suppose it were false that $\sqrt[5]{3}$ was irrational.

Suppose $\sqrt[5]{3}$ were not irrational.

(II) there would exist some integers m, n such that $n \neq 0$ and $m = n \cdot \sqrt[5]{3}$

(III) m, n have no common divisors other than $1, -1$

Alternative answer.
m, n are divisible by no integers other than $1, -1$

(IV) $m^5 = 3n^5$

(V) m^5 would be divisible by 3

(VI) a prime number

(VII) m would be divisible by 3

(VIII) there would exist some integer k such that $m = 3k$

(IX) $3^4k^5 = 3 \cdot 3^3k^5$

(X) an integer

(XI) the definition of divisibility

(XII) divisible by 3

(XIII) n would be divisible by 3

(IV) m, n had no common divisors other than $1, -1$

Alternative answer.
m, n were divisible by no integers other than $1, -1$

(XV) m, n

(c) **Solution.**

i. Write $u = \sqrt[5]{9}$, $v = \sqrt[5]{27}$, $w = \sqrt[5]{81}$.
Note that $u^3 = v^2 = (\sqrt[5]{3})^6 = 3 \cdot \sqrt[5]{3}$. Then by the result described by the statement (J), u, v are irrational.
Note that $w^4 = (\sqrt[5]{3})^{16} = 27 \cdot \sqrt[5]{3}$. Then by the result described by the statement (J), w is irrational.

ii. Pick any positive integer m. Suppose m is not divisible by 5.
By the Division Algorithm for Natural Numbers, there exists some natural numbers q, r such that $m = 5q + r$ and $0 \leq r < 5$.
Since m is not divisible by 5, we have $r \neq 0$. Then $r = 1$ or $r = 2$ or $r = 3$ or $r = 4$.
By the statement (L), and by the result in the previous part, $\sqrt[5]{3^r}$ is irrational.
Note that $\sqrt[5]{3^m} = \sqrt[5]{3^{5q+r}} = 3^q \cdot \sqrt[5]{3^r}$.
Then, by the result described by the statement (J), $\sqrt[5]{3^m}$ is irrational.

12. **Solution.**

(a) Suppose it were true that $\sqrt{2}+\sqrt{3}$ was a rational number. Write $r=\sqrt{2}+\sqrt{3}$. We have $\sqrt{3}=r-\sqrt{2}$.
Note that $3=(r-\sqrt{2})^2=r^2-2r\sqrt{2}+2$. Then $2r\sqrt{2}=r^2-1$.
Since $\sqrt{2}>0$ and $\sqrt{3}>0$, we have $r=\sqrt{2}+\sqrt{3}>0$.
Then $\dfrac{r^2-1}{2r}$ is well-defined and $\sqrt{2}=\dfrac{r^2-1}{2r}$.
By assumption, r was a rational number. Then r^2-1 and $2r$ would be rational numbers. Therefore $\dfrac{r^2-1}{2r}$ would be a rational number.
Recall that $\sqrt{2}$ is an irrational number.
Then $\sqrt{2}$ would be simultaneously a rational number and an irrational number. Contradiction arises.
It follows that $\sqrt{2}+\sqrt{3}$ is an irrational number.

(b) Suppose it were true that $\sqrt{3}-\sqrt{2}$ was a rational number. Write $s=\sqrt{3}-\sqrt{2}$. We have $\sqrt{3}=r+\sqrt{2}$.
Note that $3=(s+\sqrt{2})^2=s^2+2s\sqrt{2}+2$. Then $2s\sqrt{2}=1-s^2$.
Since $\sqrt{3}>\sqrt{2}$, we have $s=\sqrt{3}-\sqrt{2}>0$.
Then $\dfrac{1-s^2}{2s}$ is well-defined and $\sqrt{2}=\dfrac{1-s^2}{2s}$.
By assumption, s was a rational number. Then $1-s^2$ and $2s$ would be rational numbers. Therefore $\dfrac{1-s^2}{2s}$ would be a rational number.
Recall that $\sqrt{2}$ is an irrational number.
Then $\sqrt{2}$ would be simultaneously a rational number and an irrational number. Contradiction arises.
It follows that $\sqrt{2}+\sqrt{3}$ is an irrational number.

13. **Solution.**

Suppose it were true that $\sqrt{6}$ was a rational number.

Then, by the definition of rational numbers, there would exist some integers m,n such that $n\neq 0$ and $n\sqrt{6}=m$.

Without loss of generality, we assume that '$\sqrt{6}=\dfrac{m}{n}$' is a lowest-term representation as a fraction of integers.

Then m,n have no common divisor other than $1,-1$.

We would have $2(3n^2)=6n^2=m^2$.

Note that $3n^2$ is an integer. Then m^2 would be divisible by 2.

Note that 2 is a prime number.

By Euclid's Lemma, m would be divisible by 2.

There would exist some integer k such that $m=2k$.

Now $6n^2=m^2=(2k)^2=4k^2$. Then $3n^2=2k^2$.

Note that k^2 is an integer. Then $3n^2$ would be divisible by 2.

Note that 3 is a prime number. Then by Euclid's Lemma, at least one of $3,n^2$ would be divisible by 2.

Note that 3 is not divisible by 2.

Then n^2 would be divisible by 2. Again, by Euclid's Lemma, n would be divisible by 2.

Now simultaneously m,n had no common divisors other than $1,-1$, and 2 would be a common divisor of m,n. Contradiction arises.

Hence $\sqrt{6}$ is an irrational numbers in the first place.

14. **Solution.**

(a) Let p,q be positive integers. Suppose p,q are prime numbers, and q is divisible by p.
By the definition of prime numbers, q is divisible by no integers other than $1,-1,q,-q$.
Since q is divisible by p, we have $p=1$ or $p=-1$ or $p=q$ or $p=-q$.
Since p is a prime number, $p\neq -1$ and $p\neq 1$. Since p,q are positive $p\neq -q$.
Then $p=q$.

(b) Suppose p, q are distinct positive prime numbers, and m, n are positive integers.
Suppoe it were true that $p^m - q^n$ was divisible by p.
Then there would exist some integer k such that $p^m - q^n = kp$.
Therefore $q^n = p^m - kp = (p^{m-1} - k)p$.
Since p, k are integers, $p^{m-1} - k$ is also an integer. Then q^n would be divisible by p.
Then by (the generalization of) Euclid's Lemma (to the 'many-numbers situation'), q would be divisible by p.
Therefore by (♯), $p = q$. Contradiction arises.
Hence, $p^m - q^n$ is not divisible by p.
Modifying the argument above, we also deduce that $p^m - q^n$ is not divisible by q.

15. **Solution.**

(a) Let n be a positive integer. Pick any natural number x.
By the Division Algorithm for Natural Numbers, there exist some unique natural numbers q, r such that $x = nq + r$ and $0 \leq r < n$.
We verify that at least one of $x, x+1, x+2, \cdots, x+n-1$ is divisible by n:

- Suppose x is not divisible by n.
[We verify that $x + n - r$ is amongst $x, x+1, x+2, \cdots, x+n-1$ and is divisible by n.]
Since x is not divisible by n, we have $r \geq 1$. Then $n - r \leq n - 1$. Therefore $x < x + n - r \leq x + n - 1$.
Note that $x + n - r = qn + r + n - r = (q+1)n$.
Since q is a natural number, $q + 1$ is also a natural number.
Then, by the definition of divisibility, $x + n - r$ is divisible by n.

We verify that at most one of $x, x+1, x+2, \cdots, x+n-1$ is divisible by n:

- Suppose two of $x, x+1, x+2, \cdots, x+n-1$, say, $x+k, x+k'$, are divisible by n. By definition, $0 \leq k < n$ and $0 \leq k' < n$. Then $|k - k'| < n$.
By definition of divisibility, there exist some integers q, q' such that $x + k = qn$ and $x + k' = q'n$.
Note that $k - k' = (x + k) - (x + k') = qn - q'n = (q - q')n$.
Then $|q - q'|n = |k - k'| < n$
Since q, q' are integers, $|q - q'|$ is a natural number. Then $|q - q'| = 0$ or $|q - q'| \geq 1$.
(Recall that $n > 0$.) Then $|q - q'|n = 0$ or $|q - q'|n \geq n$.
Since $|k - k'| < n$, it is impossible for '$|q - q'|n \geq n$' to hold.
Therefore $|q - q'|n = 0$. (Again recall $n > 0$.) Then $|q - q'| = 0$. Therefore $q = q'$. Hence $k = k'$ also, and furthermore $x + k = x + k'$. (Why?)

It follows that exactly one of $x, x+1, x+2, \cdots, x+n-1$ is divisible by n.

(b) Suppose p is a prime number and $p \geq 5$.

i. - p is not divisible by 2. (Why?) Then p is an odd number. There exists some integer k such that $p = 2k + 1$.
Now $p^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k = 4(k+1)k$.
Exactly one of $k, k+1$ is divisible by 2. Then $(k+1)k$ is divisible by 2. (Why?) Therefore there exists some integer ℓ such that $(k+1)k = 2\ell$.
We have $p^2 - 1 = 4(k+1)k = 8\ell$. Hence $p^2 - 1$ is divisible by 8.
   - Note that $p^2 - 1 = (p-1)(p+1)$.
p is not divisible by 3. (Why?)
Exactly one of $p-1, p, p+1$ is divisible by 3. Since it cannot be p, it is exactly one of $p-1, p+1$. Then $(p-1)(p+1)$ is divisible by 3. (Why?)

ii. Since $p^2 - 1$ is divisible by 8, there exists some integer u such that $p^2 - 1 = 8u$.
Since $p^2 - 1$ is divisible by 3, there exists some integer v such that $p^2 - 1 = 3v$.
Now we have $8u = p^2 - 1 = 3v$. Then $8u$ is divisible by 3.
3 is a prime number. Then, by Euclid's Lemma, at least one of $8, u$ is divisible by 3.
Note that 8 is not divisible by 3. Then u is divisible by 3. There exists some integer w such that $u = 3w$.
Now $p^2 - 1 = 8u = 8 \cdot 3w = 24w$ and w is an integer. Therefore $p^2 - 1$ is divisible by 24.

16. (a) **Answer.**

(I) a common divisor of m, n

(II) For any integer

(III) if d is a common divisor of m, n then $|d| \leq g$

(b) **Solution.**

i. 120 is divisible by $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$, and their additive inverses, but no other integers.
75 is divisible by $1, 3, 5, 10, 15, 25, 75$, and their additive inverse, but no other integers.
The common divisors of $120, 75$ are $1, 3, 5, 10, 15$, and their additive inverses.
Hence $\gcd(120, 75) = 15$.

ii. Repeatedly applying Division Algorithm for natural numbers, we obtain these successive equalities:—

$$\begin{cases} 120 &= 1 \cdot 75 &+ 45 \\ 75 &= 1 \cdot 45 &+ 30 \\ 45 &= 1 \cdot 30 &+ 15 \\ 30 &= 2 \cdot 15 & \end{cases}$$

Hence $\gcd(120, 75) = 15$.

(c) **Solution.**

Let n be a positive integer.

Since n is a positive integer, we have $\displaystyle\sum_{j=0}^{23} n^j > \sum_{j=0}^{14} n^j > 0$.

Repeatedly applying Division Algorithm, we obtain these successive equalities:—

$$\begin{cases} \displaystyle\sum_{j=0}^{23} n^j &= n^9 \cdot \sum_{j=0}^{14} n^j &+ \displaystyle\sum_{j=0}^{8} n^j \\ \displaystyle\sum_{j=0}^{14} n^j &= n^6 \cdot \sum_{j=0}^{8} n^j &+ \displaystyle\sum_{j=0}^{5} n^j \\ \displaystyle\sum_{j=0}^{8} n^j &= n^3 \cdot \sum_{j=0}^{5} n^j &+ \displaystyle\sum_{j=0}^{2} n^j \\ \displaystyle\sum_{j=0}^{5} n^j &= (n^3+1) \cdot \sum_{j=0}^{2} n^j & \end{cases}$$

Since n is a positive integer, we indeed have the inequalities $\displaystyle\sum_{j=0}^{23} n^j > \sum_{j=0}^{14} n^j > \sum_{j=0}^{8} n^j > \sum_{j=0}^{5} n^j > \sum_{j=0}^{2} n^j > 0$.

Hence $\gcd\left(\displaystyle\sum_{j=0}^{23} n^j, \sum_{j=0}^{14} n^j\right) = n^2 + n + 1$.

17. (a) **Answer.**

Suppose a, b are integers. Then there exist some integers s, t such that $\gcd(a, b) = sa + tb$.

(b) **Solution.**

i. Let x, y, z be integers. Suppose $\gcd(y, z) = 1$. Further suppose that xy is divisible by z.
[We want to verify that there exists some integer h such that $x = hz$.]
By Bézout's Lemma, there exist some integers s, t such that $sy + tz = \gcd(y, z)$. —— $(\star)$
By assumption, $\gcd(y, z) = 1$. —— $(\star\star)$
Then by $(\star), (\star\star)$, we have $1 = sy + tz$.
Therefore $x = s \cdot xy + tx \cdot z$. —— $(\star\star\star)$
Since xy is divisible by z, there exists some integer k such that $xy = kz$.
Then by $(\star\star\star)$, we have $x = 1 \cdot x = s \cdot xy + tx \cdot z = s \cdot kz + tx \cdot z = (sk + tx)z$.
Note that s, k, t, x are integers. Then $sk + tx$ is an integer.
Then, by the definition of divisibility, x is divisible by z.

ii. Let x, y, z be integers. Suppose $\gcd(y, z) = 1$. Further suppose that x is divisible by y and x is divisible by z.
[We want to verify that there exists some integer h such that $x = h \cdot yz$.]
By the definition of divisibility, since x is divisible by y, there exists some integer g such that $x = gy$.
Also, since x is divisible by z, there exists some integer h such that $x = hz$.
By Bézout's Lemma, there exists some integers s, t such that $\gcd(y, z) = sy + tz$.
Since $\gcd(y, z) = 1$, we have $1 = \gcd(y, z) = sy + tz$.
Now $x = x \cdot 1 = x(sy + tz) = xsy + xtz = hz \cdot sy + gy \cdot tz = (hs + gt)yz$.
Therefore, by the definition of divisibility, x is divisible by yz.

(c) i. Suppose a, b be integers. Then the statements $(\natural_1), (\natural_2)$ are logically equivalent:—

A. Suppose $\gcd(a,b)=1$. Then by Bézout's Lemma, there exist some integers s,t such that $sa+tb=1$.

B. Suppose that there exist some integers s,t such that $sa+tb=1$
Note that 1 is a common divisor of a,b.
We verify the statement '*for any integer d, if d is a common divisor of a,b then $|d|\leq 1$*':—

- Pick any integer d. Suppose d is a common divisor of a,b.
Then, for the same integers s,t, the integer $sa+tb$ is divisible by d.
Therefore, there exists some integer k such that $sa+tb=kd$.
Now we have $1=sa+tb=kd=|kd|=|k|\cdot|d|$.——$(*)$
Note that $k\neq 0$. Then $|k|\geq 1$. Then by $(*)$, we have $1=|k|\cdot|d|\geq 1\cdot|d|=|d|$.

Hence, by definition, $\gcd(a,b)=1$.

ii. Let x,y,z be integers. Suppose $\gcd(x,y)=1$ and $\gcd(x,z)=1$.
[We want to verify that there exist some integers p,q such that $px+q\cdot yz=1$.]
By $(\natural)$, since $\gcd(x,z)=1$, there exist some integers s,t such that $sx+ty=1$.——$(\dagger_1)$
Also, since $\gcd(x,z)=1$, there exist some integers u,v such that $ux+vz=1$.——$(\dagger_2)$
Now by $(\dagger_1),(\dagger_2)$, we have $sux\cdot x=sx\cdot ux=(1-ty)(1-vz)=1-ty-vz+tv\cdot yz$. ——$(\dagger_3)$
Then by $(\dagger_3)$ and again by $(\dagger_1),(\dagger_2)$, we have $sux\cdot x-tv\cdot yz=1-ty-vz=(1-ty)+(1-vz)-1=sx+ux-1$.
Therefore $1=sx+ux-sux\cdot x+tv\cdot yz=(s+u-sux)\cdot x+tv\cdot yz$.
Note that s,u,x,t,v are integers. Then $s+u-sux, tv$ are also integers.
Therefore by $(\natural)$, $\gcd(x,yz)=1$.

18. **Solution.**

(a) Let u,n be integers. Suppose $n>0$.

i. [We handle the 'existence part' here.
We verify the statement '*there exist some unique integers q,r such that $u=qn+r$ and $0\leq r<n$*'.]

A. Suppose $u\geq 0$. Then by the Division Algorithm for natural numbers, there exist some unique natural numbers q,r such that $u=qn+r$ and $0\leq r<n$.
Such q,r are integers automatically.

B. Suppose $u<0$.
Write $t=|u|$. Note that t is a positive integer.
Then, by the Division Algorithm, there exist some natural numbers q',r' such that $t=q'n+r'$ and $0\leq r'<n$.
We verify that there exist some integers q,r such that $u=qn+r$ and $0\leq r<n$:

- (Case 1.) Suppose $0<r'<n$.
Define $q=-q'-1$ and $r=n-r'$. By definition, q,r are integers.
Since $0<r'<n$, we have $0<r<n$. Also, $u=-t=-q'n-r'=(-q'-1)n+(v-r')=qn+r$.
- (Case 2.) Suppose $r'=0$.
Define $q=-q'$ and $r=0$. By definition, q,r are integers.
We have $0\leq r<n$. Also, $u=-t=-q'n-r'=qn+r$.

C. [We handle the 'uniqueness part'.
We verify the statemen '*For any integers q,r,q',r', if ($u=qn+r$ and $0\leq r<n$ and $u=q'n+r'$ and $0\leq r'<n$) then $q=q'$ and $r=r'$*'.]
Let q,r,q',r' be integers. Suppose $u=qn+r$ and $0\leq r<n$ and $u=q'n+r'$ and $0\leq r'<n$.
We have $qn+r=u=q'n+r'$. Then $(q-q')n=r'-r$. (Recall $n>0$.) Therefore $|q-q'|n=|r-r'|$.
Since $0\leq r<n$ and $0\leq r'<n$, we have $|r-r'|<n$.
Therefore $0\leq|q-q'|n=|r-r'|<n$.
Note that q,q' are integers. Then $|q-q'|$ is a natural number.
Therefore $|q-q'|=0$ or $|q-q'|\geq 1$.
(Again recall $n>0$.) Hence $|q-q'|n=0$ or $|q-q'|n\geq n$.
Since $|q-q'|n=|r-r'|<n$, it is impossibe for '$|q-q'|n\geq n$' to hold.
Therefore $|q-q'|n=0$. Since $n\neq 0$, we have $|q-q'|=0$. Then $q=q'$.
We now have $|r-r'|=|q-q'|n=0$. Hence $r=r'$ also.

(b) Let x,y,p be integers. Suppose p is a positive prime number, and x is not divisible by p, and $0<y<p$.

i. [Here we handle the 'existence part'.
We verify the stateemtn '*there exists some unique integer s such that $0<s<p$ and $sx-y$ is divisible by p*'.]
Since x is not divisible by p, we have $\gcd(x,p)=1$.
By Bézout's Lemma, there exists some integers u,v such that $ux+vp=\gcd(x,p)$.

Then $ux + vp = \gcd(x, p) = 1$.
Therefore $(yu)x + vyp = y$.——(†)
(Note that yu is an integer.) By (♯), there exist some unique integers q, r such that $yu = qp + r$ and $0 \leq r < p$.
Then by (†), we have $y = (yu)x + vyp = (qp + r)x + vyp = rx + (q + v)p$. Therefore $rx - y = -(q + v)p$.
Since q, v are integers, $-(q + v)$ is also an integer. Then $rx - y$ is divisible by p.
Recall that $0 \leq r < p$. Note that $r \neq 0$; otherwise, $-y$, and also y, would be divisible by p, which is impossible (because $0 < y < p$). Then $0 < r < p$.

ii. [Here we handle the 'uniqueness part'.
We verify the stateemtn '*For any integers s, s', if ($0 < s < p$ and $sx - y$ is divisible by p and $0 < s' < p$ and $s'x - y$ is divisible by p) then $s = s'$*'.]
Let s, s' be integers. Suppose $0 < s < p$ and $sx - y$ is divisible by p and $0 < s' < p$ and $s'x - y$ is divisible by p.
Since $sx - y$ is divisible by p and $s'x - y$ is divisible by p, $(s - s')x$ is divisible by p. (Why?)
By Euclid's Lemma, at least one of $s - s', x$ is divisible by p.
By assumption x is not divisible by p. Then $s - s'$ is divisible by p. Therefore there exists some integer k such that $s - s' = kp$.
(Recall $p > 0$.) Then $|s - s'| = |kp| = |k|p$.——(‡)
Note that $|k|$ is a natural number. Then $|k| = 0$ or $|k| \geq 1$. Then $|k|p = 0$ or $|k|p \geq p$.
Since $0 < s < p$ and $0 < s' < p$, we have $|s - s'| < p$. Then by (‡), it is impossible for '$|k|p \geq p$' to hold.
Then $|k|p = 0$. (Again recall $p > 0$.) Then $k = 0$. Therefore $s = s'$.

19. **Solution.**

(a) Let a, b be complex numbers. Suppose $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$.
Further suppose it were true that both of a, b were zero.
Then we would have $a^4 + a^3b + a^2b^2 + ab^3 + b^4 = 0^4 + 0^3 \cdot 0 + 0^2 \cdot 0^2 + 0 \cdot 0^3 + 0^4 = 0$.
However, by assumption, $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$. Then $0 \neq 0$. Contradiction arises.
Hence at least one of a, b is non-zero in the first place.

(b) Let a, b be a real number and b be a complex number. Suppose $a^3|b| > 2$.
Further suppose it were true that $a^6 + 9|b|^2 \leq 6$.
Since $a^3|b| > 2$, we have $-3a^3|b| < -6$.
Therefore $(a^3 - 3|b|)^2 = a^6 + 9|b|^2 - 6a^3|b| < 6 - 6 = 0$.
Note that $|b|$ is a real number. Since $a, |b|$ are real numbers, $a^3 - 3|b|$ is a real number. Then $(a^3 - 3|b|)^2 \geq 0$.
Then $0 \leq (a^3 - 3|b|)^2 < 0$. Contradiction arises.
Hence $a^6 + 9|b|^2 > 6$ in the first place.

(c) Let ζ be a complex number. Suppose that for any positive real number ε, the inequality $|\zeta| \leq \varepsilon$ holds.
Further suppose it were true that $\zeta \neq 0$.
Since $\zeta \neq 0$, we would have $|\zeta| \neq 0$. Then $|\zeta| > 0$.
Define $\varepsilon = \dfrac{|\zeta|}{2}$. Since $|\zeta| > 0$ and $\dfrac{1}{2} > 0$, we would have $\varepsilon > 0$.
Then by assumption, $|\zeta| \leq \varepsilon = \dfrac{|\zeta|}{2}$.
Therefore $\dfrac{|\zeta|}{2} = |\zeta| - \dfrac{|\zeta|}{2} \leq 0$. Hence $|\zeta| = 2 \cdot \dfrac{|\zeta|}{2} \leq 0$ (because $2 > 0$).
Now we have $0 < |\zeta| \leq 0$. Contradiction arises.
Hence $\zeta = 0$ in the first place.

(d) Let a, b be real numbers. Suppose $a > b > 0$.
Further suppose that it were true that $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} \leq a$.
Note that $\sqrt{a^2 - b^2} \geq 0$ and $\sqrt{2ab - b^2} \geq 0$. Then $a \geq \sqrt{a^2 - b^2} + \sqrt{2ab - b^2} \geq 0$.——($\sharp_1$)
Since $a > b > 0$, we have $a^2 - b^2 = (a - b)(a + b) \geq 0$. Then $(\sqrt{a^2 - b^2})^2 = a^2 - b^2$.——($\sharp_2$)
Similarly, since $a > b > 0$, we have $2ab - b^2 = (2a - b)b \geq 0$. Then $(\sqrt{2ab - b^2})^2 = 2ab - b^2$.——($\sharp_3$)
Therefore we would have

$$\begin{aligned} a^2 &\geq \left(\sqrt{a^2 - b^2} + \sqrt{2ab - b^2}\right)^2 \quad (\text{by } (\sharp_1)) \\ &= (a^2 - b^2) + (2ab - b^2) + 2\sqrt{(a^2 - b^2)(2ab - b^2)} \quad (\text{by } (\sharp_2), (\sharp_3)) \\ &= a^2 - 2b^2 + 2ab + 2\sqrt{(a - b)(a + b)(2a - b)b}. \end{aligned}$$

Hence $0 \le \sqrt{(a-b)(a+b)(2a-b)b} \le b^2 - ab = b(b-a)$.

Recall that by assumption, $a > b > 0$. Then $b > 0$ and $b - a < 0$. Therefore $b(b-a) < 0$.

Hence $0 \le b(b-a) < 0$.

Contradiction arises.

Hence, in the first place, $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} > a$.

20. **Solution.**

For each positive integer n, define $A_n = \sum_{j=1}^{n} \frac{1}{j}$, $B_n = \sum_{k=1}^{n} \frac{1}{2k}$, $C_n = \sum_{k=1}^{n} \frac{1}{2k-1}$.

(a) i. Suppose n is a positive integer.

Then $B_n = \sum_{j=1}^{n} \frac{1}{2j} = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{j} = \frac{1}{2} A_n$.

Now $A_{2n} - \frac{1}{2} A_n = A_{2n} - B_n = \sum_{j=1}^{2n} \frac{1}{j} - \sum_{k=1}^{n} \frac{1}{2k} = \sum_{k=1}^{n} \frac{1}{2k-1} = C_n$.

ii. Suppose n is an integer greater than 1.

Whenever $j = 2, 3, \cdots, n$, we have $\frac{1}{2j-1} - \frac{1}{2j} = \frac{1}{2j(2j-1)} \ge 0$.

Then

$$\begin{aligned} C_n - B_n &= \sum_{j=1}^{n} \frac{1}{2j-1} - \sum_{j=1}^{n} \frac{1}{2j} \\ &= \frac{1}{2 \cdot 1 - 1} + \sum_{j=2}^{n} \frac{1}{2j-1} - \frac{1}{2 \cdot 1} - \sum_{j=2}^{n} \frac{1}{2j} \\ &= \frac{1}{2} + \sum_{j=2}^{n} \left(\frac{1}{2j-1} - \frac{1}{2j} \right) \ge \frac{1}{2}. \end{aligned}$$

(b) Suppose it were true that $\{A_n\}_{n=1}^{\infty}$ converged in $\mathbb{R}$. Write $\lim_{n\to\infty} A_n = \ell$.

Then the infinite sequence $\{A_{2n}\}_{n=1}^{\infty}$ would converge in $\mathbb{R}$ also, and $\lim_{n\to\infty} A_{2n} = \ell$.

Since $B_n = \frac{1}{2} A_n$ for each positive integer n, the infinite sequence $\{B_n\}_{n=1}^{\infty}$ would converge in $\mathbb{R}$. Moreover

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \frac{1}{2} A_n = \frac{1}{2} \lim_{n\to\infty} A_n = \frac{\ell}{2}.$$

Since $C_n = A_{2n} - \frac{1}{2} A_n$ for any positive integer n, the infinite sequence $\{C_n\}_{n=1}^{\infty}$ would converge in $\mathbb{R}$. Moreover,

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \left(A_{2n} - \frac{1}{2} A_n \right) = \lim_{n\to\infty} A_{2n} - \frac{1}{2} \lim_{n\to\infty} A_n = \frac{\ell}{2}.$$

Then the infinite sequence $\{C_n - B_n\}_{n=1}^{\infty}$ would converge in $\mathbb{R}$. Moreover,

$$\lim_{n\to\infty} (C_n - B_n) = \lim_{n\to\infty} C_n - \lim_{n\to\infty} B_n = \frac{\ell}{2} - \frac{\ell}{2} = 0.$$

However, since $C_n - B_n \ge \frac{1}{2}$ for each positive integer $n \in \mathbb{N}$, we have $\lim_{n\to\infty} (C_n - B_n) \ge \frac{1}{2}$.

(Then $0 = \lim_{n\to\infty} (C_n - B_n) \ge \frac{1}{2}$.) Contradiction arises.

Hence, in the first place, $\{A_n\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}$.