MATH1050BC/1058 Assignment 2 (Answers and selected solution)

1. **Answer.**

(a) True. (c) True. (e) True. (g) False. (i) False.

(b) False. (d) False. (f) False. (h) False. (j) False.

2. **Answer.**

(a)
(I) Suppose $x + y > 1$ and $x > y$
(II) $(x - y)(x + y - 1)$
(III) Since
(IV) $x > y$
(V) $x + y - 1 > 0$ and $x - y > 0$
(VI) $(x - y)(x + y) - (x - y) > 0$
(VII) $x^2 - y^2 > x - y$

(b)
(I) Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$.
(II) $(x + y)(x^2 - xy + y^2) - xy(x + y) = (x + y)(x^2 - 2xy + y^2) = (x + y)(x - y)^2$
(III) $x - y$ is (also) a real number
(IV) ≥ 0
(V) $(x^3 + y^3) - xy(x + y) \geq 0$

(c)
(I) Suppose $x^2 + y^2 + z^2 + xy - yz - xz \leq 0$.
(II) 0
(III) $\left(y^2 + z^2 - 2yz\right) + \left(x^2 + z^2 - 2xz\right)$
(IV) $(x + y)^2 + (y - z)^2 + (x - z)^2$
(V) $(y - z)^2 \geq 0$ and $(x - z)^2 \geq 0$
(VI) $(x + y)^2 + (y - z)^2 + (x - z)^2$
(VII) 0
(VIII) $(y - z)^2 = 0$ and $(x - z)^2 = 0$
(IX) $x + y = 0$ and $y - z = 0$ and $x - z = 0$
(X) $x = 0$ and $y = 0$
(XI) $z = 0$

(d)
(I) Suppose $y > x > 0$ and $z > -y$
(II) $z > -y$
(III) > 0
(IV) Suppose
(V) $zy > zx$
(VI) $\dfrac{(x + z)y - x(y + z)}{y(y + z)} > 0$
(VII) Suppose $\dfrac{x + z}{y + z} > \dfrac{x}{y}$
(VIII) $\dfrac{x + z}{y + z} \cdot y(y + z) > \dfrac{x}{y} \cdot y(y + z)$
(IX) $zy - zx = (xy + zy) - (xy + zx) > 0$
(X) and
(XI) $z < 0$ and $y - x < 0$
(XII) $\dfrac{x + z}{y + z} > \dfrac{x}{y}$ iff $z > 0$

3. **Solution.**

(a) Let x be a real number. Suppose $2x - \frac{3}{x} \geq 1$. —— $(\star)$

Note that $x \neq 0$ (because $\frac{3}{x}$ is well-defined as a real number.)

Since x is a non-zero real number, $x^2 > 0$. Then by $(\star)$, we have

$$2x^3 - 3x = \left(2x - \frac{3}{x}\right) \cdot x^2 =\geq 1 \cdot x^2 = x^2$$

Therefore $x(x+1)(2x-3) = 2x^3 - x^2 - 3x \geq 0$.

Hence $-1 \leq x \leq 0$ or $x \geq 1.5$.

Recall that $x \neq 0$. Then $-1 \leq x < 0$ or $x \geq 1.5$.

(b) Let x be a real number. Suppose $\frac{x^2-1}{x^2-4} \leq -2$ —— $(\star)$.

Note that $x^2 - 4 \neq 0$ (because $\frac{x^2-1}{x^2-4}$ is well-defined as a real number). Then $x \neq -2$ and $x \neq 2$.

Since $x^2 - 4 \neq 0$, $x^2 - 4 > 0$.

Then, by $(\star)$, we have

$$(x^2-1)(x^2-4) = \frac{x^2-1}{x^2-4} \cdot (x^2-4)^2 \leq -2(x^2-4)^2.$$

Therefore

$$3(x+2)(x+\sqrt{3})(x-\sqrt{3})(x-2) = (x^2-4)(3x^2-9) = (x^2-4)[(x^2-1)+2(x^2-4)] \leq 0.$$

Hence $-2 \leq x \leq -\sqrt{3}$ or $\sqrt{3} \leq x \leq 2)$.

Recall that $x \neq -2$ and $x \neq 2$. Then $-2 < x \leq -\sqrt{3}$ or $\sqrt{3} \leq x < 2$.

(c) Let x be a real number. Suppose $|x^2 - 5x| < 6$.

Then $-6 < x^2 - 5x < 6$. ——$(\star)$

In particular, we have $x^2 - 5x > -6$ by $(\star)$.

Then $(x-2)(x-3) = x^2 - 5x + 6 > 0$.

Therefore $x < 2$ or $x > 3$. —— $(\star_1)$

Also by $(\star)$, we have $x^2 - 5x < 6$.

Then $(x+1)(x-6) = x^2 - 5x - 6 < 0$.

Therefore $-1 < x < 6$. —— $(\star_2)$

Now, by $(\star_1), (\star_2)$ (simultaneously), we have ($x < 2$ or $x > 3$) and $-1 < x < 6$.

Then (by Distributive Law for conjunction and disjunction), we have

$$(x < 2 \text{ and } -1 < x < 6) \text{ or } (x > 3 \text{ and } -1 < x < 6)$$

Therefore $-1 < x < 2$ or $3 < x < 6$.

(d) Let x be a real number. Suppose $\left|\frac{3x+11}{x+2}\right| < 2$. —— $(\star)$

Note that $x \neq -2$ (because $\frac{3x+11}{x+2}$ is well-defined as a real number.)

Since $x \neq -2$, $|x+2| > 0$.

By $(\star)$, we have

$$0 \leq |3x+11| = \left|\frac{3x+11}{x+2}\right| \cdot |x+2| = \frac{|3x+11|}{|x+2|} \cdot |x+2| < 2|x+2|.$$

Then

$$(3x+11)^2 = |3x+11|^2 < (2|x+2|)^2 = 4(x+2)^2.$$

Therefore

$$5(x+3)(x+7) = [(3x+11)+2(x+2)][(3x+11)-2(x+2)] = (3x+11)^2 - 4(x+2)^2 < 0$$

Hence $-7 < x < -3$. (Note that '$x \neq -2$' has been incorporated also.)

(e) Let x be a real number. Suppose $|\ |x| - 4\ | > 3$. ——($\star$)

We have $|x| - 4 < -3$ or $|x| - 4 > 3$.

- (Case 1.) Suppoe $|x| - 4 < -3$. Then $|x| < 1$. Therefore $-1 < x < 1$.
- (Case 2.) Suppose $|x| - 4 > 3$. Then $|x| > 7$. Therefore $x < -7$ or $x > 7$.

Hence, in any case, $x < -7$ or $-1 < x < 1$ or $x > 7$.

(f) Let x be a real number. Suppose $\dfrac{|x^2-3|}{(x-1)^2 \cdot \sqrt{x+2}} \leq \dfrac{2|x|}{(x-1)^2 \cdot \sqrt{x+2}}$. —— ($\star$)

Note that $x \neq -1$ and $x > -2$ (because $\dfrac{|x^2-3|}{(x-1)^2 \cdot \sqrt{x+2}}, \dfrac{2|x|}{(x-1)^2 \cdot \sqrt{x+2}}$ are well-defined as real numbers.)

Since $x \neq -1$, $(x-1)^2 > 0$.

Since $x > -2$, $\sqrt{x+2} > 0$.

By ($\star$), we have

$$(x^2-3)^2 = |x^2-3|^2 = \frac{|x^2-3|}{(x-1)^2 \cdot \sqrt{x+2}} \cdot (x-1)^2 \cdot \sqrt{x+2} \leq \frac{2|x|}{(x-1)^2 \cdot \sqrt{x+2}} \cdot (x-1)^2 \cdot \sqrt{x+2} = (2|x|)^2 = 4x^2.$$

Then

$$(x-3)(x+1)(x-1)(x+3) = (x^2-2x-3)(x^2+2x-3) = [(x^2-3)-2x][(x^2-3)+2x] = (x^2-3)^2 - 4x^2 \leq 0$$

Then $-3 \leq x \leq -1$ or $1 \leq x \leq 3$.

Recall that $x \neq -1$ and $x > -2$. Then Then $-2 < x \leq -1$ or $1 < x \leq 3$.

4. **Solution.**

(a) Let x, y be real numbers. Suppose $x < y < 1$.

Note that $\dfrac{y}{1-y} - \dfrac{x}{1-x} = \dfrac{y(1-x) - x(1-y)}{(1-x)(1-y)} = \dfrac{y-x}{(1-x)(1-y)}$. —— ($\star$)

Since $x < 1$, we have $1 - x > 0$. Since $y < 1$, we have $1 - y > 0$.

Since $x < y$, we have $y - x > 0$.

Since $1 - x > 0$ and $1 - y > 0$ and $y - x > 0$, we have $\dfrac{y-x}{(1-x)(1-y)} > 0$.

Then by ($\star$), we have $\dfrac{y}{1-y} - \dfrac{x}{1-x} > 0$.

Therefore $\dfrac{x}{1-x} < \dfrac{y}{1-y}$.

(b) *Argument with the help of the previous part.*

Let x, y be real numbers. Suppose $0 < x < y < 1$.

By ($\sharp$), we have $\dfrac{x}{1-x} < \dfrac{y}{1-y}$. —— ($\dagger$)

By assumption, we have $x > 0$. Then by ($\dagger$) also, we have $\dfrac{x^2}{1-x} = x \cdot \dfrac{x}{1-x} < x \cdot \dfrac{y}{1-y}$. —— ($\ddagger$)

By assumption, we have $0 < y < 1$. Then $\dfrac{y}{1-y} > 0$.

Therefore by ($\ddagger$) also, we have $x \cdot \dfrac{y}{1-y} < y \cdot \dfrac{y}{1-y} = \dfrac{y^2}{1-y}$.

We now have $\dfrac{x^2}{1-x} < x \cdot \dfrac{y}{1-y}$ and $x \cdot \dfrac{y}{1-y} < \dfrac{y^2}{1-y}$.

Then $\dfrac{x^2}{1-x} < \dfrac{y^2}{1-y}$.

Direct argument.

Let x, y be real numbers. Suppose $0 < x < y < 1$.

Note that $\dfrac{y^2}{1-y} - \dfrac{x^2}{1-x} = \dfrac{y^2(1-x) - x^2(1-y)}{(1-x)(1-y)} = \cdots\cdots\cdots = \dfrac{(y-x)[1-(1-x)(1-y)]}{(1-x)(1-y)}$. —— ($\star$)

Since $0 < x < 1$, we have $0 < 1 - x < 1$. Since $0 < y < 1$, we have $0 < 1 - y < 1$.

Now we have $0 < 1 - x < 1$ and $0 < 1 - y < 1$. Then $0 < (1-x)(1-y) < 1$. Therefore $1 - (1-x)(1-y) > 0$.

Since $x < y$, we have $y - x > 0$.

Since $1 - x > 0$ and $1 - y > 0$ and $y - x > 0$ and $1 - (1-x)(1-y) > 0$, we have $\dfrac{(y-x)[1-(1-x)(1-y)]}{(1-x)(1-y)} > 0$.

Then by $(\star)$, we have $\dfrac{y^2}{1-y} - \dfrac{x^2}{1-x} > 0$.

Therefore $\dfrac{x^2}{1-x} < \dfrac{y^2}{1-y}$.

5. **Solution.**

(a) Let x be a real number. Suppose $x > 0$.
Note that $\sqrt{x+2}, \sqrt{x+1}, \sqrt{x}$ are well-defined and $(\sqrt{x+2})^2 = x+2$, $(\sqrt{x+1})^2 = x+1$ and $(\sqrt{x})^2 = x$.
We have $\sqrt{x+2} + \sqrt{x+1} > 2\sqrt{x+1} > 0$. Therefore

$$\begin{aligned}\sqrt{x+2} - \sqrt{x+1} = \frac{(\sqrt{x+2}-\sqrt{x+1})(\sqrt{x+2}+\sqrt{x+1})}{\sqrt{x+2}+\sqrt{x+1}} &= \frac{(x+2)-(x+1)}{\sqrt{x+2}+\sqrt{x+1}} \\ &= \frac{1}{\sqrt{x+2}+\sqrt{x+1}} \\ &< \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}\end{aligned}$$

We also have $0 < \sqrt{x+1} + \sqrt{x} < 2\sqrt{x+1}$. Then

$$\begin{aligned}\sqrt{x+1} - \sqrt{x} = \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} &= \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x+1}+\sqrt{x}} \\ &> \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}\end{aligned}$$

Hence $\sqrt{x+2} - \sqrt{x+1} < \dfrac{1}{2\sqrt{x+1}} < \sqrt{x+1} - \sqrt{x}$.

(b) Let k be an integer amongst $10, 11, 12, \cdots, 10000$.
We have $2(\sqrt{k+1} - \sqrt{k}) < \dfrac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1})$.
Then

$$2(\sqrt{10001} - \sqrt{10}) = \sum_{k=10}^{10000} 2(\sqrt{k+1} - \sqrt{k}) < \sum_{k=10}^{10000} \frac{1}{\sqrt{k}} < \sum_{k=10}^{10000} 2(\sqrt{k} - \sqrt{k-1}) = 2(\sqrt{10000} - \sqrt{9})$$

Note that $2(\sqrt{10000} - \sqrt{9}) = 2(100 - 3) = 194$.
Also note that $2(\sqrt{10001} - \sqrt{10}) > 2(\sqrt{10000} - \sqrt{3.3^2}) = 2(100 - 3.3) = 193.4 > 193$.
Therefore $193 < \displaystyle\sum_{k=10}^{10000} \frac{1}{\sqrt{k}} < 194$.

6. **Solution.**
Suppose x, y are real numbers.
We have

$$\begin{aligned}(x^2+y^2)^3 - (x^3+y^3)^2 &= (x^6 + 3x^4y^2 + 3x^2y^4 + y^6) - (x^6 + 2x^3y^3 + y^6) \\ &= x^2y^2(3x^2 + 3y^2 - 2xy) \\ &= x^2y^2[2x^2 + 2y^2 + (x-y)^2] \quad \text{——}(\dagger)\end{aligned}$$

Since x, y are real numbers, $(x-y)^2 \geq 0$.
Note that $x^2 \geq 0$ and $y^2 \geq 0$.
Then $x^2y^2[2x^2 + 2y^2 + (x-y)^2] \geq 0$.

Therefore, by (†), we have $(x^2+y^2)^3-(x^3+y^3)^2 \geq 0$.

Hence $(x^2+y^2)^3 \geq (x^3+y^3)^2$.

We now verify that $(x^2+y^2)^3 = (x^3+y^3)^2$ iff $x=0$ or $y=0$:—

- Suppose $x=0$ or $y=0$.
  - Suppose $x=0$. Then $(x^2+y^2)^3 = y^6 = (x^3+y^3)^2$.
  - Suppose $y=0$. Then $(x^2+y^2)^3 = x^6 = (x^3+y^3)^2$.
- Suppose $(x^2+y^2)^3 = (x^3+y^3)^2$. Then, by (†), we have $x^2y^2[2x^2+2y^2+(x-y)^2]=0$.
  Therefore $x=0$ or $y=0$ or $2x^2+2y^2+(x-y)^2=0$.
  If $2x^2+2y^2+(x-y)^2=0$ then $x=y=x-y=0$.
  Hence $x=0$ or $y=0$.

7. **Solution.**

Let m,n be positive integers. Let x be a positive real number. Suppose $m>n$.

We have

$$\left(x^m+\frac{1}{x^m}\right)-\left(x^n+\frac{1}{x^n}\right)=(x^m-x^n)-\frac{x^m-x^n}{x^{m+n}}=\frac{(x^m-x^n)(x^{m+n}-1)}{x^{m+n}}.$$

Note that $x^{m+n}>0$. We verify that $(x^m-x^n)(x^{m+n}-1)\geq 0$:

- (Case 1). Suppose $0<x<1$. Then $0<x^{m-n}<1$ and $x^n>0$.
  Therefore $x^m-x^n=(x^{m-n}-1)x^n<0$. Hence $x^m-x^n<0$.
  Also $0<x^{m+n}<1$. Then $x^{m+n}-1<0$.
  It follows that $(x^m-x^n)(x^{m+n}-1)>0$.
- (Case 2). Suppose $x\geq 1$. Then $x^{m-n}\geq 1$ and $x^n>0$.
  Therefore $x^m-x^n=(x^{m-n}-1)\cdot x^n>0$. Hence $x^m-x^n\geq 0$.
  Also $x^{m+n}\geq 1$. Then $x^{m+n}-1\geq 0$.
  It follows that $(x^m-x^n)(x^{m+n}-1)\geq 0$.

Therefore, in any case, $(x^m-x^n)(x^{m+n}-1)\geq 0$.

It follows that $\left(x^m+\frac{1}{x^m}\right)-\left(x^n+\frac{1}{x^n}\right)=\frac{(x^m-x^n)(x^{m+n}-1)}{x^{m+n}}\geq 0$. Hence $x^m+\frac{1}{x^m}\geq x^n+\frac{1}{x^n}$.

- Suppose $x=1$. Then $x^m+\frac{1}{x^m}=2=x^n+\frac{1}{x^n}$.
- Suppose $x^m+\frac{1}{x^m}=x^n+\frac{1}{x^n}$. Then

$$0=\left(x^m+\frac{1}{x^m}\right)-\left(x^n+\frac{1}{x^n}\right)=\frac{(x^m-x^n)(x^{m+n}-1)}{x^{m+n}}.$$

  Therefore $x^m=x^n$ or $x^{m+n}=1$.
  - (Case 1). Suppose $x^m=x^n$. Then $x^{m-n}=1$. Since $m>n$, we have $m-n>0$. Since $x>0$, we have $x=1$.
  - (Case 2). Suppose $x^{m+n}=1$. Note that $m+n>0$. Since $x>0$, we have $x=1$.

  Hence, in any case, we have $x=1$.

8. **Solution.**

(a) Suppose u,v are positive real numbers.
$\sqrt{u}$, $\sqrt{v}$, $\sqrt{uv}$ are well-defined, and $u=(\sqrt{u})^2$, $v=(\sqrt{v})^2$, $\sqrt{uv}=\sqrt{u}\sqrt{v}$.
$\sqrt{u}-\sqrt{v}$ is well-defined as a real number. Then $(\sqrt{u}-\sqrt{v})^2\geq 0$.
Therefore $u+v=(\sqrt{u})^2+(\sqrt{v})^2=(\sqrt{u}-\sqrt{v})^2+2\sqrt{u}\sqrt{v}\geq 2\sqrt{u}\sqrt{v}=2\sqrt{uv}$.
Hence $\frac{u+v}{2}\geq\sqrt{uv}$.

(b) Suppose a, b, c, d are positive real numbers.

Then $a + b \geq 2\sqrt{ab}$.

Also, $c + d \geq 2\sqrt{cd}$.

Note that $\sqrt{ab}, \sqrt{cd}$ are also positive real numbers. Then $\dfrac{\sqrt{ab}+\sqrt{cd}}{2} \geq \sqrt{\sqrt{ab}\sqrt{cd}}$.

Therefore,

$$\frac{a+b+c+d}{4} = \frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right) \geq \frac{\sqrt{ab}+\sqrt{cd}}{2} = \sqrt{(\sqrt{ab})\cdot(\sqrt{cd})} = \sqrt{\sqrt{abcd}} = \sqrt[4]{abcd}.$$

(c) Suppose r, s, t are positive real numbers. Define $u = \dfrac{r+s+t}{3}$. Note that u is also a positive real number.

We have $\dfrac{r+s+t+u}{4} \geq \sqrt[4]{rstu}$.

Note that $\dfrac{r+s+t+u}{4} = \dfrac{r+s+t+(r+s+t)/3}{4} = \dfrac{r+s+t}{3} = u$.

Then $u = \dfrac{r+s+t+u}{4} \geq \sqrt[4]{rstu} = \sqrt[4]{rst}\cdot\sqrt[4]{u}$.

Note that $u > 0$, and $\sqrt[4]{u} > 0$. Then $\left(\sqrt[4]{u}\right)^3 \geq \sqrt[4]{rst}$.

Therefore $\dfrac{r+s+t}{3} = u = \left[\sqrt[3]{\left(\sqrt[4]{u}\right)^3}\right]^4 \geq \left(\sqrt[3]{\sqrt[4]{rst}}\right)^4 = \left(\sqrt[12]{rst}\right)^4 = \sqrt[3]{rst}$.

9. (a) **Answer.**

Suppose u, v are real numbers. Then $|u+v| \leq |u| + |v|$. Moreover, equality holds iff $uv \geq 0$.

(b) **Answer.**

i. Suppose κ, λ are complex numbers. Then we say κ is a non-negative scalar multiple of λ if there exists a non-negative real number c such that $\kappa = c\lambda$.

ii. Suppose ζ, η be complex numbers. Then $|\zeta + \eta| \leq |\zeta| + |\eta|$. Moreover, equality holds iff at least one of ζ, η is a non-negative scalar multiple of the other.

(c) **Solution.**

i. Let x, a, b be real numbers. Suppose $b > 0$, and $|x - a| < b$. —— $(\star)$

We have

$$\begin{aligned} |x^2 - a^2| &= |(x-a)(x+a)| = |(x-a)[(x-a)+2a]| = |x-a|\cdot|(x-a)+2a| \\ &\leq |x-a|(|x-a| + |2a|) \quad \text{(by Triangle Inequality)} \\ &= |x-a|(|x-a| + 2|a|) \\ &< b(b+2|a|) \quad \text{(by } (\star)\text{)} \end{aligned}$$

ii. Let x, a, b be real numbers. Suppose $b > 0$, and $|x - a| < b$. —— $(\star)$

We have

$$\begin{aligned} |x^3 - a^3| &= |[(x-a)+a]^3 - a^3| = |(x-a)^3 + 3a(x-a)^2 + 3a^2(x-a)| \\ &\leq |(x-a)^3| + |3a(x-a)^2| + |3a^2(x-a)| \quad \text{(by Triangle Inequality)} \\ &= |x-a|^3 + 3|a|\cdot|x-a|^2 + 3a^2\cdot|x-a| \\ &< b^3 + 3|a|b^2 + 3a^2 b = b(b^2 + 3|a|b + 3a^2) \quad \text{(by } (\star)\text{)} \end{aligned}$$

iii. Let $\zeta, \alpha, \beta, \gamma$ be complex numbers. Suppose $|\overline{\zeta} + 2i\alpha| \leq 1$, $|\zeta - \beta/2| \leq 1$, $|\zeta - 4\gamma| \leq 3$. —— $(\star)$

Then

$$\begin{aligned} |2\zeta - 2i\overline{\alpha} - \beta + 4\gamma| &= |(\zeta - 2i\overline{\alpha}) + (2\zeta - \beta) + (-\zeta + 4\gamma)| \\ &\leq |\zeta - 2i\overline{\alpha}| + |2\zeta - \beta| + |-\zeta + 4\gamma| \quad \text{(by Triangle Inequality)} \\ &= |\overline{(\zeta - 2i\overline{\alpha})}| + |2(\zeta - \beta/2)| + |-(\zeta - 4\gamma)| \\ &= |\overline{\zeta} + 2i\alpha| + 2|\zeta - \beta/2| + |\zeta - 4\gamma| \\ &\leq 1 + 2\cdot 1 + 3 = 6 \quad \text{(by } (\star)\text{)} \end{aligned}$$

iv. Let ζ be a complex number. Suppose $0 < |\zeta| < 1$.
Note that $1 - |\zeta| > 0$. —— $(\star_1)$
Also note that $|\zeta|^{1050} > 0$. —— $(\star_2)$
Further note that $0 < |\zeta|^{3011} < 1$. Then $0 < 1 - |\zeta|^{3011} < 1$. —— $(\star_3)$
We have

$$\begin{aligned}
\left|\sum_{k=1050}^{4060} \zeta^k\right| &\leq \sum_{k=1050}^{4060} |\zeta^k| \quad \text{(by Triangle Inequality)} \\
&= \sum_{k=1050}^{4060} |\zeta|^k \\
&= |\zeta|^{1050} \cdot \sum_{k=0}^{3010} |\zeta|^k \\
&= |\zeta|^{1050} \cdot \frac{1 - |\zeta|^{3011}}{1 - |\zeta|} \\
&< \frac{|\zeta|^{1050}}{1 - |\zeta|} \quad \text{(by } (\star_1), (\star_2), (\star_3))
\end{aligned}$$

v. Let α be a complex number and n be a positive integer. Suppose $|\alpha| > 5$.
(Note that $\alpha \neq 0$. So $\dfrac{1}{\alpha}$ is well-defined as a complex number. Also note that $\alpha \neq 5$. So $\dfrac{1}{\alpha - 5}, \dfrac{1}{1 - 5/\alpha}$ are well-defined as complex numbers.)
We have

$$\sum_{k=0}^{n} \frac{5^k}{\alpha^k} = \sum_{k=0}^{n} \left(\frac{5}{\alpha}\right)^k = \frac{1 - (5/\alpha)^{n+1}}{1 - 5/\alpha} = \frac{1 - 5^{n+1}/\alpha^{n+1}}{1 - 5/\alpha}.$$

Then $\displaystyle\sum_{k=0}^{n} \frac{5^k}{\alpha^k} - \frac{\alpha}{\alpha - 5} = \sum_{k=0}^{n} \frac{5^k}{\alpha^k} - \frac{1}{1 - 5/\alpha} = -\frac{5^{n+1}/\alpha^{n+1}}{1 - 5/\alpha} = -\frac{5^{n+1}}{\alpha^n(\alpha - 5)}$.
By assumption, $|\alpha| > 5$. Then by the Triangle Inequality, we have $|\alpha - 5| \geq |\alpha| - 5 > 0$. —— $(\star)$
Therefore

$$\begin{aligned}
\left|\sum_{k=0}^{n} \frac{5^k}{\alpha^k} - \frac{\alpha}{\alpha - 5}\right| &= \left|-\frac{5^{n+1}}{\alpha^n(\alpha - 5)}\right| \\
&= \frac{5^{n+1}}{|\alpha|^n |\alpha - 5|} \\
&\leq \frac{5^{n+1}}{|\alpha|^n (|\alpha| - 5)} \quad \text{(by } (\star))
\end{aligned}$$

10. **Solution.**

(a) Suppose n is an integer greater than 2.

Note that for each integer k between 2 and n, we have $0 < \dfrac{1}{n} < 1$, $0 < \dfrac{2}{n} < 1$, ..., $0 < \dfrac{k-1}{n} < 1$.

Then, for the same k, the inequalities below hold:—

$$0 < \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) < 1. \quad \text{——}(\dagger)$$

Then

$$
\begin{aligned}
\left(1+\frac{2}{n}\right)^n &= \sum_{j=0}^{n} \binom{n}{j} \cdot \left(\frac{2}{n}\right)^j = 1 + n \cdot \frac{2}{n} + \sum_{j=2}^{n} \frac{n(n-1)(n-2)\cdot \ldots \cdot (n-j+1)}{j!} \cdot \frac{2^j}{n^j} \\
&= 3 + \sum_{j=2}^{n} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdot \ldots \cdot \left(1-\frac{j-1}{n}\right) \cdot \frac{2^j}{j!} \\
&\leq 1 + \frac{2}{1!} + \sum_{j=2}^{n} 1 \cdot \frac{2^j}{j!} \quad \text{——}(\dagger) \\
&= \sum_{j=0}^{n} \frac{2^j}{j!}
\end{aligned}
$$

(b) Suppose n is an integer greater than 2.

Note that $\left(1+\frac{2}{n}\right)^n > 0$, and $\left(1+\frac{2}{n+1}\right)^{n+1} > 0$.

We have

$$
\begin{aligned}
\frac{\left(1+\frac{2}{n+1}\right)^{n+1}}{\left(1+\frac{2}{n}\right)^n} &= \left[\frac{\left(1+\frac{2}{n+1}\right)}{\left(1+\frac{2}{n}\right)}\right]^n \cdot \left(1+\frac{2}{n+1}\right) = \left[\frac{n(n+3)}{(n+1)(n+2)}\right]^n \cdot \left(1+\frac{2}{n+1}\right) \\
&= \left[1 - \frac{2}{(n+1)(n+2)}\right]^n \cdot \left(1+\frac{2}{n+1}\right) \quad \text{——}(\ddagger_1)
\end{aligned}
$$

Note that $-1 < \frac{2}{(n+1)(n+2)} < 0$.

Then, by Bernoulli's Inequality, we have

$$
\left[1 - \frac{2}{(n+1)(n+2)}\right]^n > 1 - \frac{2n}{(n+1)(n+2)} \quad \text{——}(\ddagger_1)
$$

Since $1 + \frac{2}{n+1} > 0$, we have by $(\ddagger_1), (\ddagger_2)$,

$$
\begin{aligned}
\frac{\left(1+\frac{2}{n+1}\right)^{n+1}}{\left(1+\frac{2}{n}\right)^n} &= \left[1 - \frac{2}{(n+1)(n+2)}\right]^n \cdot \left(1+\frac{2}{n+1}\right) \\
&> \left[1 - \frac{2n}{(n+1)(n+2)}\right]\left(1+\frac{2}{n+1}\right) = 1 + \frac{2}{n+1} - \frac{2n}{(n+1)(n+2)} - \frac{4n}{(n+1)^2(n+2)} \\
&= 1 + \frac{2(n+1)(n+2) - 2n(n+1) - 4n}{(n+1)^2(n+2)} = 1 + \frac{4}{(n+1)^2(n+2)} > 1
\end{aligned}
$$

Recall that $\left(1+\frac{2}{n}\right)^n > 0$. Then $\left(1+\frac{2}{n}\right)^n < \left(1+\frac{2}{n+1}\right)^{n+1}$.

11. **Answer.**

(a) (I) Suppose

(II) $t^4 - s^4 = (t^2 - s^2)(t^2 + s^2) = (t-s)(t+s)(t^2+s^2)$

(III) $s < t$

(IV) $s \geq 0$ and

(V) $s + t > 0$

(VI) $t^2 > 0$

(VII) $s \geq 0$

(VIII) $t^2 + s^2 > 0$

(IX) $f(t) - f(s) > 0$

(X) f is strictly increasing on $[0, +\infty)$

(b) (I) Suppose f is strictly decreasing on $\mathbb{R}$.

(II) Pick any $s, t \in \mathbb{R}$. Suppose $s < t$.

(III) $g(s) - g(t) = (f(s) - 2s^3) - (f(t) - 2t^3) = (f(s) - f(t)) + 2(t - s)(t^2 + st + s^2)$

(IV) Since f is strictly decreasing and $s < t$

(V) $t^2 + st + s^2$

(VI) 0

(VII) $f(s) - f(t) + 2(t - s)(t^2 + st + s^2) > 0$

(VIII) $g(s) > g(t)$

12. **Solution.**

(a) Denote by $P(n)$ the proposition below:—

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

- [We verify $P(1)$.]
  We have $1 \leq 1 = 2\sqrt{1} - 1$. Hence $P(1)$ is true.
- [We verify the statement '*for any positive integer k, if $P(k)$ is true then $P(k+1)$ is true*'.]
  Let k be a positive integer. Suppose $P(k)$ is true.
  Then $(2\sqrt{k} - 1) - \left(1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{k}}\right) \geq 0$.
  We deduce that $P(k+1)$ is true:
  We have

$$\begin{aligned}
& (2\sqrt{k+1} - 1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right) \\
= \;& (2\sqrt{k+1} - 1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right) \\
= \;& 2(\sqrt{k+1} - \sqrt{k}) - \frac{1}{\sqrt{k+1}} + (2\sqrt{k} - 1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}\right) \\
\geq \;& 2(\sqrt{k+1} - \sqrt{k}) - \frac{1}{\sqrt{k+1}} \quad \text{(by } P(k)\text{)} \\
= \;& \frac{2}{\sqrt{k+1} + \sqrt{k}} - \frac{1}{\sqrt{k+1}} \\
= \;& \frac{\sqrt{k+1} - \sqrt{k}}{(\sqrt{k+1} + \sqrt{k})\sqrt{k+1}} = \frac{1}{(\sqrt{k+1} + \sqrt{k})^2\sqrt{k+1}} \\
\geq \;& 0
\end{aligned}$$

  Then $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{k}} + \dfrac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$. Hence $P(k+1)$ is true .

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

(b) Denote by $P(n)$ the proposition $\dfrac{(3n)!}{(n!)^3} > \dfrac{27^n}{9n}$.

- [We verify $P(1)$.]
  We have $\dfrac{(3 \cdot 1)!}{(1!)^3} = 6 > 3 = \dfrac{27^1}{9 \cdot 1}$.
- [We verify the statement '*for any positive integer k, if $P(k)$ is true then $P(k+1)$ is true*'.]
  Let k be a positive integer. Suppose $P(k)$ is true.
  Then $\dfrac{(3k)!}{(k!)^3} \cdot \dfrac{9k}{27^k} > 1$. ——($\sharp$)
  We deduce that $P(k+1)$ is true:—

$$\begin{aligned}
\frac{[3(k+1)]!}{[(k+1)!]^3}\cdot\frac{9(k+1)}{27^{k+1}} &= \frac{(3k+3)(3k+2)(3k+1)}{(k+1)^3}\cdot\frac{(3k)!}{(k!)^3}\cdot\frac{9k}{27^k}\cdot\frac{k+1}{27k}\\
&= \frac{(3k)!}{(k!)^3}\cdot\frac{9k}{27^k}\cdot\frac{(k+2/3)(k+1/3)}{(k+1)k}\\
&> 1\cdot\frac{(k+2/3)(k+1/3)}{(k+1)k} \quad (\text{by } (\sharp))\\
&= \frac{k^2+k+2/9}{k^2+k} = 1+\frac{2}{9k^2+9k}\\
&> 1 \quad \left(\text{because } \frac{2}{9k^2+9k}>0\right)
\end{aligned}$$

(Note that $27^{k+1}>0$ and $9(k+1)>0$.) Then $\frac{[3(k+1)]!}{[(k+1)!]^3} > \frac{27^{k+1}}{9(k+1)}$.

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

13. (a) **Solution.**

Suppose A, B, a, b are real numbers.

Then $(A^2+a^2)(B^2+b^2)-(AB+ab)^2 = A^2B^2+a^2b^2+A^2b^2+B^2a^2-A^2B^2-a^2b^2-2ABab = A^2b^2+B^2a^2-2ABab = (Ab-Ba)^2$.

Sine A, B, a, b are real numbers, $Ab-Ba$ is a real number. Then $(Ab-Ba)^2 \geq 0$.

Therefore $(A^2+a^2)(B^2+b^2)-(AB+ab)^2 \geq 0$.

Hence $(A^2+B^2)(a^2+b^2) \geq (Aa+Bb)^2$.

(b) **Answer.**

(I) If $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers, then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_jb_j\right)^2$.

Alternative answer.

Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers. Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_jb_j\right)^2$.

Unacceptable (because they are wrong).

- For any integer n greater than 1, if $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers, then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_jb_j\right)^2$.
- $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_jb_j\right)^2$.

(II) For any integer k greater than 1, if $P(k)$ is true then $P(k+1)$ is true.

(III) Let k be an integer greater than 1. Suppose $P(k)$ is true.

(IV) Suppose $a_1, a_2, \cdots, a_k, a_{k+1}, b_1, b_2, \cdots, b_k, b_{k+1}$ are non-negative real numbers.

(V) By $P(2)$, we have $\left(\sum_{j=1}^{k+1} {a_j}^2\right)\left(\sum_{j=1}^{k+1} {b_j}^2\right) = \left(A^2+{a_{k+1}}^2\right)\left(B^2+{b_{k+1}}^2\right) \geq \left(AB+a_{k+1}b_{k+1}\right)^2$. ——($\dagger_1$)

By $P(k)$, we have $A^2B^2 = \left(\sum_{j=1}^{k} {a_j}^2\right)\left(\sum_{j=1}^{k} {b_j}^2\right) \geq \left(\sum_{j=1}^{k} a_jb_j\right)^2 = C^2$.

Since A, B, C are non-negative real numbers, we have $AB \geq C$.

Then $AB + a_{k+1}b_{k+1} \geq C + a_{k+1}b_{k+1} = \sum_{j=1}^{k+1} a_jb_j \geq 0$.

Therefore $\left(AB+a_{k+1}b_{k+1}\right)^2 \geq \left(\sum_{j=1}^{k+1} a_jb_j\right)^2$. ——($\dagger_2$)

Hence by $(\dagger_1), (\dagger_2)$, we have $\left(\sum_{j=1}^{k+1} {a_j}^2\right)\left(\sum_{j=1}^{k+1} {b_j}^2\right) \geq (AB + a_{k+1}b_{k+1})^2 \geq \left(\sum_{j=1}^{k+1} a_j b_j\right)^2$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any integer n greater than 1.

(c) **Solution.**

Suppose m is a positive integer.

For each $j = 1, 2, \cdots, m, m+1, \cdots, 2m-1, 2m$, define $a_j = \frac{1}{j}$, and $b_j = \frac{1}{2m-j+1}$.

Note that $a_j = b_{2m-j}$.

So $\sum_{j=1}^{2m} {a_j}^2 = \sum_{j=1}^{2m} {b_j}^2 = \sum_{j=1}^{2m} \frac{1}{j^2}$. —— $(\dagger)$

Also,

$$\begin{aligned}\sum_{j=1}^{2m} a_j b_j &= \sum_{j=1}^{m} a_j b_j + \sum_{j=m+1}^{2m} a_j b_j = 2\sum_{j=1}^{m} a_j b_j \\ &= 2\left[\frac{1}{1\cdot 2m} + \frac{1}{2(2m-1)} + \frac{1}{3(2m-2)} + \cdots + \frac{1}{k(2m-k+1)} + \cdots + \frac{1}{m(m+1)}\right] \quad \text{——}(\ddagger)\end{aligned}$$

Note that $a_1, a_2, \cdots, a_m, a_{m+1}, \cdots, a_{2m-1}, a_{2m}, b_1, b_2, \cdots, b_m, b_{m+1}, \cdots, b_{2m-1}, b_{2m}$ are non-negative real numbers. We have

$$\begin{aligned}&\left\{2\left[\frac{1}{1\cdot 2m} + \frac{1}{2(2m-1)} + \frac{1}{3(2m-2)} + \cdots + \frac{1}{k(2m-k+1)} + \cdots + \frac{1}{m(m+1)}\right]\right\}^2 \\ = &\left(\sum_{j=1}^{2m} a_j b_j\right)^2 \qquad \text{(by } (\ddagger)) \\ \leq &\left(\sum_{j=1}^{2m} {a_j}^2\right)\left(\sum_{j=1}^{2m} {b_j}^2\right) \qquad \text{(by } (\sharp)) \\ = &\left(\sum_{j=1}^{2m} \frac{1}{j^2}\right)^2 \qquad \text{(by } (\dagger))\end{aligned}$$