

Advice.

1. The questions in this assignment are mainly about inequalities. Do familiarize yourself with the corresponding material available in the course homepage before trying the questions.
2. All questions are about manipulating and reasoning with inequalities, and many very often involve various kinds of ‘algebraic techniques’. A few involve the application of mathematical induction. But first of all you must pay attention on the behaviour of the symbols ‘<’, ‘>’, ‘≤’, ‘≥’, which is not the same as that of the symbol ‘=’.
3. Questions which require more thought and/or work and/or tricks and/or organization and/or ... than the ‘unlabelled’ questions are labelled by $\diamond, \clubsuit, \heartsuit, \spadesuit$ in ascending order of overall difficulty level.

Instructions.

1. Any work submitted by you must be written on **A4-size sheets** and must be appropriately binded.
Your name and student ID, as in your student card, and the **code** of the section to which you are registered must be written at the upper right corner of the first page of your submission.
2. (a) **Mandatory work, for assessment purpose.**
 You are **required** to submit work on Questions (1), (2), (3a), (3d), (9a), (9b), (9c.i), (9c.iii), (11), (12), (13a), (13b) for course assessment purpose.
 (b) **Optional proof-writing exercise.**
 You may also **opt** to submit, **on exactly one sheet, separate from your submission on mandatory work**, your work on Question (4a). It will be read and commented, but not counted for course assessment.
 (c) **Other optional work.**
 You may choose submit work on other questions in this assignment not mentioned above, alongside the mandatory work, but there is no guarantee that it will be read.
3. (a) You must adhere to the notations which have been in the course. Things which have not been formally defined in the course are not allowed in your work.
 (b) You do not need to use anything from calculus (limits, derivatives, integrals) for your arguments. You are not allowed to use anything from calculus for your arguments.
 (c) When you are giving a proof for something, you must make it clear what you are assuming (and indicate your assumptions with the use of appropriate words at appropriate places). You are also expected to indicate the ‘flow of logic’ in the argument with correct and appropriate use of words/symbols.
 (d) Words of the likes of ‘trivial’, ‘obvious’, ‘clear(-ly)’ are not allowed to appear in your work.

* * *

1. *In this question, you are not required to give any justification to your answers.*

Which of the statements below are true? Which are false?

- (a) Let x, y be real numbers. Suppose $x < y$. Then $x \leq y$.
- (b) Let x, y be real numbers. Suppose $x \leq y$. Then $x < y$.
- (c) Let x, y be real numbers. Suppose $x < y$. Then $2x - 1 < 2y + 3$.
- (d) Let x, y be real numbers. Suppose $x < y$. Then $2x < 3y$.
- (e) Let x, y be real numbers. Suppose $x^2 < y^2$. Then $x^4 < y^4$.
- (f) Let x, y be real numbers. Suppose $x^2 < y^2$. Then $x^3 < y^3$.
- (g) Let x, y be non-zero real numbers. Suppose $x < y$. Then $\frac{1}{x} < \frac{1}{y}$.
- (h) Let x, y be non-zero real numbers. Suppose $x < y$. Then $\frac{1}{x} > \frac{1}{y}$.
- (i) Let x, y, z be real numbers. Suppose $y > 2x$ and $z > 3x$. Then $z > y$.
- (j) Let x, y, z be real numbers. Suppose $y > 2x$ and $z > 3x$. Then $y > z$.

Remark. Prove each of those statements that you believe to be true. Provide a counter-example against each of those statements (and hence a dis-proof against the statement concerned) that you believe to be false.

2.◇ Denote by (A) , (B) , (C) , (D) the respective statements below:—

- (A) Let $x, y \in \mathbb{R}$. Suppose $x + y > 1$ and $x > y$. Then $x^2 - y^2 > x - y$.
 (B) Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$. Then $x^3 + y^3 \geq xy(x + y)$.
 (C) Let $x, y, z \in \mathbb{R}$. Suppose $x^2 + y^2 + z^2 + xy - yz - xz \leq 0$. Then $x = y = z = 0$.
 (D) Let $x, y, z \in \mathbb{R}$. Suppose $y > x > 0$ and $z > -y$. Then $\frac{x+z}{y+z} > \frac{x}{y}$ iff $z > 0$.

For each statement:—

- **either** (1) fill in the blanks (all labelled by capital-letter Roman numerals) in its *partially completed* proof in the corresponding block below, with appropriate words/symbols so as to obtain a *complete* proof for the statement,
- **or** (2) write your own proof for the statement concerned.

(Note that the ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

- (a) This is a partially completed proof for the statement (A) . (*Complete it, or write your own proof.*)

Let $x, y \in \mathbb{R}$. _____ (I) .
 We note that $(x - y)(x + y) - (x - y) =$ _____ (II) . — (★)
 _____ (III) $x + y > 1$, we have $x + y - 1 > 0$.
 Since _____ (IV) , we have $x - y > 0$.
 Since _____ (V) , we have $(x - y)(x + y - 1) > 0$.
 Then, by (★), _____ (VI) .
 Therefore _____ (VII) .

- (b) This is a partially completed proof for the statement (B) . (*Complete it, or write your own proof.*)

_____ (I) .
 We have $(x^3 + y^3) - xy(x + y) =$ _____ (II) . — (★)
 Since $x > 0$ and $y > 0$, we have $x + y > 0$.
 Since x, y are real numbers, _____ (III) . Then $(x - y)^2 \geq 0$.
 Since $x + y > 0$ and $(x - y)^2 \geq 0$, we have $(x + y)(x - y)^2$ _____ (IV) .
 Then, by (★), _____ (V) .
 Hence $x^3 + y^3 \geq xy(x + y)$.

- (c) This is a partially completed proof for the statement (C) . (*Complete it, or write your own proof.*)

Let $x, y, z \in \mathbb{R}$. _____ (I) .
 By assumption, we have $2(x^2 + y^2 + z^2 + xy - yz - xz) \leq$ _____ (II) . — (★)
 Note that $2(x^2 + y^2 + z^2 + xy - yz - xz) = (x^2 + y^2 + 2xy) +$ _____ (III) = _____ (IV) . — (★★)
 Since x, y, z are real numbers, each of $x + y, y - z, x - z$ is a real number.
 Then $(x + y)^2 \geq 0$ and _____ (V) respectively. — (★★★)
 Now by (★), (★★), (★★★), we have $0 \geq$ _____ (VI) $\geq (x + y)^2 \geq$ _____ (VII) .
 Then $(x + y)^2 = 0$.
 Modifying the argument above, we also deduce _____ (VIII) .
 Therefore _____ (IX) .
 Since $y - z = 0$ and $x - z = 0$, we have $x = y = z$.
 Since $x + y = 0$ and $x = y$, we have _____ (X) . Then, since $x = z$, we have _____ (XI) also.

- (d) This is a partially completed proof for the statement (D) . (*Complete it, or write your own proof.*)

Let $x, y, z \in \mathbb{R}$. _____ (I) .

Since _____ (II) , we have $y + z > 0$. Then, since $y > 0$ also, we have $y(y + z)$ _____ (III) .

- [We want to deduce: ‘If $z > 0$ then $\frac{x+z}{y+z} > \frac{x}{y}$.’]

_____ (IV) $z > 0$.

Then, since $z > 0$ and $y > x$, we have _____ (V) .

Therefore $(x + z)y = xy + zy > xy + zx = x(y + z)$.

Then $\frac{x+z}{y+z} - \frac{x}{y} =$ _____ (VI) .

Therefore $\frac{x+z}{y+z} > \frac{x}{y}$.

- [We want to deduce: ‘If $\frac{x+z}{y+z} > \frac{x}{y}$ then $z > 0$.’]

_____ (VII) .

Then $xy + zy = (x + z)y =$ _____ (VIII) $= x(y + z) = xy + zx$.

Therefore $z(y - x) =$ _____ (IX) .

Then ($z > 0$ _____ (X) $y - x > 0$) or (_____ (XI)).

Since $y > x$, we have $y - x > 0$.

Hence $z > 0$ and $y - x > 0$. In particular, $z > 0$.

It follows that _____ (XII) .

3. Prove the statements below, (with reference to the properties of strict/non-strict inequalities regarding the arithmetic operations $+$, $-$, $\times$, $\div$, and the absolute value $|\cdot|$):—

(a) Let x be a real number. Suppose $2x - \frac{3}{x} \geq 1$. Then $-1 \leq x < 0$ or $x \geq 1.5$.

(b) Let x be a real number. Suppose $\frac{x^2 - 1}{x^2 - 4} \leq -2$. Then $-2 < x \leq -\sqrt{3}$ or $\sqrt{3} \leq x < 2$.

(c) Let x be a real number. Suppose $|x^2 - 5x| < 6$. Then $-1 < x < 2$ or $3 < x < 6$.

(d) Let x be a real number. Suppose $\left| \frac{3x + 11}{x + 2} \right| < 2$. Then $-7 < x < -3$.

(e) Let x be a real number. Suppose $||x| - 4| > 3$. Then $x < -7$ or $-1 < x < 1$ or $x > 7$.

(f)[◇] Let x be a real number. Suppose $\frac{|x^2 - 3|}{(x - 1)^2 \cdot \sqrt{x + 2}} \leq \frac{2|x|}{(x - 1)^2 \cdot \sqrt{x + 2}}$. Then $-2 < x \leq -1$ or $1 < x \leq 3$.

Remark. For each statement above, what is its converse? Is its converse true? If yes, can you prove it?

4. (a) Prove the statement (♯):

(♯) Let x, y be real numbers. Suppose $x < y < 1$. Then $\frac{x}{1 - x} < \frac{y}{1 - y}$.

(b) Prove the statement (♭):

(♭) Let x, y be real numbers. Suppose $0 < x < y < 1$. Then $\frac{x^2}{1 - x} < \frac{y^2}{1 - y}$.

Remark. There are two possible approaches. One is to make a clever and careful use of the previous part. The other is a direct argument (which involves a careful analysis of the algebra).

5.♣ In this question you may take for granted the validity of the statement (★):

(★) Let u, v be positive real numbers. Suppose $u > v$. Then $\sqrt{u} > \sqrt{v}$.

(a) Prove the statement (♯):

(#) Let x be a real number. Suppose $x > 0$. Then $\sqrt{x+2} - \sqrt{x+1} < \frac{1}{2\sqrt{x+1}} < \sqrt{x+1} - \sqrt{x}$.

Remark. You may need to make use of the ‘rationalization formula’ $\sqrt{u} - \sqrt{v} = \frac{u-v}{\sqrt{u} + \sqrt{v}}$. (But be careful on whether the formula is indeed valid with what you ‘substitute’ into u, v .)

(b) Hence prove that $193 < \sum_{k=10}^{10000} \frac{1}{\sqrt{k}} < 194$.

Remark. You may need to make use of the ‘telescopic formula’ $(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_p - u_{p-1}) = u_p - u_0$. You may also need to use the inequality $3.3^2 > 10$.

6. $\diamond$ Prove the statement (#):

(#) Suppose x, y are real numbers. Then $(x^2 + y^2)^3 \geq (x^3 + y^3)^2$. Moreover, equality holds iff $(x = 0 \text{ or } y = 0)$.

7. $\clubsuit$ Prove the statement (#):

(#) Let m, n be positive integers. Let x be a positive real number. Suppose $m > n$. Then $x^m + \frac{1}{x^m} \geq x^n + \frac{1}{x^n}$. Moreover, equality holds iff $x = 1$.

8. (a) By considering the non-negativity of squares of real numbers, or otherwise, prove the statement (‡) below:

(‡) Suppose u, v are positive real numbers. Then $\frac{u+v}{2} \geq \sqrt{uv}$.

(b) $\diamond$ Hence, or otherwise, prove the statement (#) below:

(#) Suppose a, b, c, d are positive real numbers. Then $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

(c) $\clubsuit$ Hence, or otherwise, prove the statement (b) below:

(b) Suppose r, s, t be positive real numbers. Then $\frac{r+s+t}{3} \geq \sqrt[3]{rst}$.

Remark. These are ‘baby versions’ of the Arithmetico-Geometrical Inequality.

9. (a) State (without proof) the *Triangle Inequality on the real line*.

(b) i. Explain the phrase ‘non-negative scalar multiple of a complex number’ by giving an appropriate definition.
ii. State (without proof) the *Triangle Inequality on the complex plane*.

(c) By applying the Triangle Inequality, or otherwise, prove the statements below:

i. $\diamond$ Let x, a, b be real numbers. Suppose $b > 0$, and $|x - a| < b$. Then $|x^2 - a^2| < b(b + 2|a|)$.

ii. $\clubsuit$ Let x, a, b be real numbers. Suppose $b > 0$, and $|x - a| < b$. Then $|x^3 - a^3| < b(b^2 + 3|a|b + 3a^2)$.

iii. $\diamond$ Let $\zeta, \alpha, \beta, \gamma$ be complex numbers. Suppose $|\bar{\zeta} + 2i\alpha| \leq 1$ and $|\zeta - \beta/2| \leq 1$ and $|\zeta - 4\gamma| \leq 3$. Then $|2\zeta - 2i\bar{\alpha} - \beta + 4\gamma| \leq 6$.

iv. $\clubsuit$ Let ζ be a complex number. Suppose $0 < |\zeta| < 1$. Then $\left| \sum_{k=1050}^{4060} \zeta^k \right| < \frac{|\zeta|^{1050}}{1 - |\zeta|}$.

v. $\heartsuit$ Let α be a complex number and n be a positive integer. Suppose $|\alpha| > 5$.

Then $\left| \sum_{k=0}^n \frac{5^k}{\alpha^k} - \frac{\alpha}{\alpha - 5} \right| \leq \frac{5^{n+1}}{|\alpha|^n(|\alpha| - 5)}$.

10. (a) $\clubsuit$ By applying Binomial Theorem, or otherwise, prove the statement (#):—

(#) Suppose n is an integer greater than 2. Then $\left(1 + \frac{2}{n}\right)^n \leq \sum_{j=0}^n \frac{2^j}{j!}$.

(b) $\clubsuit$ By applying Bernoulli’s Inequality, or otherwise, prove the statement (‡):—

(‡) Suppose n is an integer greater than 2. Then $\left(1 + \frac{2}{n}\right)^n < \left(1 + \frac{2}{n+1}\right)^{n+1}$.

Remark. The results (#), (‡) are instrumental to showing that $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$, $\lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{2^j}{j!}$ exist and are equal to each other (and to the number ‘ e^2 ’).

11.◇ We introduce/recall the definitions on *strict monotonicity* for real-valued functions of one real variable:

Let I be an interval. Suppose $h : D \rightarrow \mathbb{R}$ is a real-valued function of one real variable with domain D which contains I as a subset entirely. Then:—

- We say h is **strictly increasing** on I if the statement (StrIncr) holds:
(StrIncr) For any $s, t \in I$, if $s < t$ then $h(s) < h(t)$.
- We say h is **strictly decreasing** on I if the statement (StrDecr) holds:
(StrDecr) For any $s, t \in I$, if $s < t$ then $h(s) > h(t)$.

Fill in the blanks (all labelled by capital-letter Roman numerals) in the blocks below, with appropriate words and symbols so that they give respectively a proof for the statement (E) and a proof for the statement (F). (The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.)

- (a) Define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^4$ for any $x \in \mathbb{R}$. For such a function f , we prove the statement (E):
(E) f is strictly increasing on the interval $[0, +\infty)$.

[We are going to verify the statement (†): ‘For any $s, t \in [0, +\infty)$, if $s < t$ then $f(s) < f(t)$.’]

Pick any $s, t \in [0, +\infty)$. (I) $s < t$.

[We want to deduce $f(t) - f(s) > 0$.]

We have $f(t) - f(s) =$ (II) . — (★)

[We want to check that each of $t - s$, $t + s$, $t^2 + s^2$ is positive. First we ask whether ‘ $t - s > 0$ ’ is true.]

Since (III) , we have $t - s > 0$.

[Next we ask whether it is true that $t + s > 0$.]

Since (IV) $s < t$, we have $t > 0$. Then, since $s \geq 0$ and $t > 0$, we have (V) .

[Finally we ask whether it is true that $t^2 + s^2 > 0$.]

Since $t > 0$, we have (VI) . Since (VII) , we have $s^2 \geq 0$. Then (VIII) .

Now, since $t - s > 0$ and $t + s > 0$ and $t^2 + s^2 > 0$, we have $(t - s)(t + s)(t^2 + s^2) > 0$.

Then by (★), we have (IX) . Therefore $f(s) < f(t)$.

It follows from definition that (X) .

- (b) Here we prove the statement (F):

(F) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = f(x) - 2x^3$ for any $x \in \mathbb{R}$. Suppose f is strictly decreasing on $\mathbb{R}$. Then g is strictly decreasing on $\mathbb{R}$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = f(x) - 2x^3$ for any $x \in \mathbb{R}$.
(I) _____

[We are going to verify (possibly with the help of the assumption ‘ f is strictly decreasing on $\mathbb{R}$ ’) the statement (†): ‘For any $s, t \in \mathbb{R}$, if $s < t$ then $g(s) > g(t)$.’]

(II) _____

[We want to deduce $g(s) - g(t) > 0$.]

We have (III) . — (★)

(IV) _____ , we have $f(s) - f(t) > 0$.

Since $s < t$, we have $t - s > 0$. Also, (V) $= \frac{1}{2}[t^2 + (t + s)^2 + s^2] \geq$ (VI) .

Then $2(t - s)(t^2 + st + s^2) \geq 0$. Since $f(s) - f(t) > 0$ and $2(t - s)(t^2 + st + s^2) \geq 0$, we have (VII) .

Then by (★), we have $g(s) - g(t) > 0$. Therefore (VIII) .

It follows from definition that g is strictly decreasing on $\mathbb{R}$.

12. Apply mathematical induction to prove the statements below:—

- (a) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$ for any positive integer n .
- (b) $\frac{(3n)!}{[(n)!]^3} > \frac{27^n}{9n}$ for any positive integer n .

13. (a) Prove the statement (★):—

(★) Suppose A, B, a, b are real numbers. Then $(A^2 + a^2)(B^2 + b^2) \geq (AB + ab)^2$.

(b)♣ Consider the statement (♯):—

(♯) Let n be an integer greater than 1. Suppose $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are non-negative real numbers.

$$\text{Then } \left(\sum_{j=1}^n a_j^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \geq \left(\sum_{j=1}^n a_j b_j \right)^2.$$

Fill in the blanks (all labelled by capital-letter Roman numerals) in the block below, with appropriate passages so that it gives an argument for the statement (♯) by **mathematical induction**.

Denote by $P(n)$ the proposition below:

(I)

- By (★), $P(2)$ is true.
- [We want to verify this statement: For any integer k greater than 1, _____ (II) .]

(III)

We verify that $P(k+1)$ is true below:—

(IV)

$$\text{Define } A = \sqrt{\sum_{j=1}^k a_j^2}, B = \sqrt{\sum_{j=1}^k b_j^2}, C = \sum_{j=1}^k a_j b_j.$$

(Note that $\sum_{j=1}^k a_j^2, \sum_{j=1}^k b_j^2$ are non-negative real numbers. So A, B are well-defined as non-negative real numbers.

Also note that by assumption, C is a non-negative real number.)

(V)

Hence $P(k+1)$ is true indeed.

(VI)

(c)♣ By applying the result above, or otherwise, prove the statement (‡).

(‡) For any positive integer m , the inequality

$$2 \left[\frac{1}{1 \cdot 2m} + \frac{1}{2(2m-1)} + \cdots + \frac{1}{k(2m-k+1)} + \cdots + \frac{1}{m(m+1)} \right] \leq \sum_{j=1}^{2m} \frac{1}{j^2}$$

holds.