MATH1050BC/1058 Assignment 1 (Answers and selected solution)

1. **Solution.**

   Let $\omega = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$.

   (a) i. $\omega = \cos(\dfrac{\pi}{6}) + i\sin(\dfrac{\pi}{6})$.

   ii. $\omega^2 = \cos(\dfrac{\pi}{3}) + i\sin(\dfrac{\pi}{3}) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.
   $\omega^3 = \cos(\dfrac{\pi}{2}) + i\sin(\dfrac{\pi}{2}) = i$.
   $\omega^5 = \cos(\dfrac{5\pi}{6}) + i\sin(\dfrac{5\pi}{6}) = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$.
   $\omega^6 = \cos(\pi) + i\sin(\pi) = -1$.
   $\omega^{11} = \cos(\dfrac{11\pi}{6}) + i\sin(\dfrac{11\pi}{6}) = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$.
   $\omega^{12} = \cos(2\pi) + i\sin(2\pi) = 1$.

   (b) (Note that $\omega^{1050}, \omega^{1051}, \omega^{1052}, \cdots, \omega^{2229+1050}, \omega^{2230+1050}$ form a geometric progression with initial term ω^{1050} and with common ratio ω.)

$$\begin{aligned}
\sum_{k=0}^{2230} \omega^{k+1050} &= \omega^{1050} \cdot \sum_{k=0}^{2230} \omega^k \\
&= \omega^{1050} \cdot \frac{1-\omega^{2231}}{1-\omega} \\
&= \omega^6 \cdot \frac{1-\omega^{11}}{1-\omega} \\
&= \omega^6 \cdot \frac{\omega \cdot \omega^{-1} - \omega^{12} \cdot \omega^{-1}}{1-\omega} \\
&= \omega^5 \cdot \frac{\omega - 1}{1-\omega} \\
&= -\omega^5 = \frac{\sqrt{3}}{2} - \frac{1}{2}i
\end{aligned}$$

2. **Solution.**

   Let α, β, γ be complex numbers. Suppose $|\alpha| = 3$, $|\beta| = \dfrac{1}{4}$ and $|\gamma| = 5$.

   (a) $\left|\dfrac{i \cdot \alpha^3 \cdot \overline{\beta} \cdot \overline{\gamma^3}}{\overline{\alpha} \cdot \beta^2 \cdot \gamma^2}\right| = \dfrac{|i| \cdot |\alpha|^3 \cdot |\beta| \cdot |\gamma|^3}{|\alpha| \cdot |\beta|^2 \cdot |\gamma|^2} = \dfrac{1 \cdot 3^3 \cdot (1/4) \cdot 5^3}{3 \cdot (1/4)^2 \cdot 5^2} = 180$.

   (b) It is further supposed that $\mathsf{Re}(\alpha) = 2$, $\mathsf{Im}(\dfrac{1}{\beta^2}) = 14$, and $\mathsf{Re}(i\gamma^3) = 100$.

   i. $|\mathsf{Im}(\alpha)|^2 = \sqrt{|\alpha|^2 - (\mathsf{Re}(\alpha))^2} = \sqrt{3^2 - 2^2} = \sqrt{5}$.

   ii. $\left|\mathsf{Re}(\dfrac{1}{\beta^2})\right| = \sqrt{\left|\dfrac{1}{\beta^2}\right|^2 - \left(\mathsf{Im}(\dfrac{1}{\beta^2})\right)^2} = \sqrt{\dfrac{1}{|\beta|^4} - \left(\mathsf{Im}(\dfrac{1}{\beta^2})\right)^2} = \sqrt{4^4 - 14^2} = 2\sqrt{15}$.

   iii. Note that $\dfrac{\overline{\gamma}^2}{\gamma} = \dfrac{\overline{\gamma}^3}{\gamma\overline{\gamma}} = \dfrac{\overline{\gamma}^3}{|\gamma|^2} = \dfrac{\overline{\gamma}^3}{25}$.
   Also note that $i\gamma^3 = \mathsf{Re}(i\gamma^3) + i\mathsf{Im}(i\gamma^3)$. Then $\gamma^3 = \mathsf{Im}(i\gamma^3) - i\mathsf{Re}(i\gamma^3)$. (The real and imaginary parts of $i\gamma^3$ are $\mathsf{Im}(i\gamma^3)$, $-\mathsf{Re}(i\gamma^3)$ respectively.)
   Therefore $\dfrac{\overline{\gamma}^3}{25} = \dfrac{\overline{\gamma^3}}{25} = \dfrac{1}{25}(\overline{\mathsf{Im}(i\gamma^3) - i\mathsf{Re}(i\gamma^3)}) = \dfrac{1}{25}\mathsf{Im}(i\gamma^3) + i \cdot \dfrac{1}{25}\mathsf{Re}(i\gamma^3)$.
   Hence $\mathsf{Im}(\dfrac{\overline{\gamma}^2}{\gamma}) = \mathsf{Im}(\dfrac{\overline{\gamma}^3}{25}) = \dfrac{1}{25}\mathsf{Re}(i\gamma^3) = \dfrac{1}{25} \cdot 100 = 4$.

3. **Solution.**

(a) Suppose ζ, η are complex numbers. Then we say that ζ is a square root of η if $\eta = \zeta^2$.

(b) Let r be a positive real number, and θ be a real number. Suppose $\zeta = r(\cos(\theta) + i\sin(\theta))$.
There are two square roots for the number ζ.
They are: $\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$, $-\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$.
Alternative answer. The two square roots for ζ are: $\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$, $\sqrt{r}(\cos(\theta/2 + \pi) + i\sin(\theta/2 + \pi))$.

(c) Let α be a non-zero complex number. Suppose $\alpha^5/\overline{\alpha}^3 = 4$. Also suppose that α is neither real nor purely imaginary.

i. Note that $4 = |4| = |\alpha^5/\overline{\alpha}^3| = |\alpha|^5/|\overline{\alpha}|^3 = \dfrac{|\alpha|^5}{|\alpha|^3} = |\alpha|^2$.
Then $|\alpha| = 2$.

ii. We have $4 = \alpha^5/\overline{\alpha}^3 = \dfrac{\alpha^8}{|\alpha|^6} = \dfrac{\alpha^8}{64}$. Then $\alpha^8 = 256$.
Therefore $\alpha^4 = 16$ or $\alpha^4 = -16$.
Hence $\alpha^2 = 4$ or $\alpha^2 = -4$ or $\alpha^2 = 4i$ or $\alpha^2 = -4i$.
Since α is neither real nor purely imaginary, the possibilities '$\alpha^2 = 4$', '$\alpha^2 = -2$' are rejected.
Therefore $\alpha^2 = 4i$ or $\alpha^2 = -4i$.

- (Case 1.) Suppose $\alpha^2 = 4i$.
Note that $4i = 4(\cos(\pi/2) + i\sin(\pi/2))$.
Then $\alpha = 2(\cos(\pi/4) + i\sin(\pi/4))$ or $\alpha = -2(\cos(\pi/4) + i\sin(\pi/4))$.
- (Case 2.) Suppose $\alpha^2 = -4i$.
Note that $-4i = 4(\cos(3\pi/2) + i\sin(3\pi/2))$.
Then $\alpha = 2(\cos(3\pi/4) + i\sin(3\pi/4))$ or $\alpha = -2(\cos(3\pi/4) + i\sin(3\pi/4))$.

Hence $\alpha = \sqrt{2} + \sqrt{2}i$ or $\alpha = \sqrt{2} - \sqrt{2}i$ or $\alpha = -\sqrt{2} + \sqrt{2}i$ or $\alpha = -\sqrt{2} - \sqrt{2}i$.

4. **Answer.**

(a) 2, $4 + 2i$.

(b) i. 5.
ii. $5 - i$.

(c) i. $2 + 2i$, $4i$
ii. $\sqrt{2}$
iii. $-1 + i$

5. **Answer.**

(a) Suppose ζ, η are complex numbers. Suppose n is a positive integer. Then we say ζ is an n-th root of η if $\zeta^n = \eta$.

(b) i. There are exactly five quintic roots of $32i$. They are:—

- $2(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$,
- $2(\cos(\frac{9\pi}{10}) + i\sin(\frac{9\pi}{10}))$,
- $2(\cos(-\frac{7\pi}{10}) + i\sin(-\frac{7\pi}{10}))$,
- $2(\cos(-\frac{3\pi}{10}) + i\sin(-\frac{3\pi}{10}))$,
- $2(\cos(\frac{\pi}{10}) + i\sin(\frac{\pi}{10}))$.

ii. $z = 2(\cos(\frac{\pi}{10}) + i\sin(\frac{\pi}{10})$ or $z = 2(\cos(-\frac{3\pi}{10}) + i\sin(-\frac{3\pi}{10})$ or $z = 2(\cos(-\frac{7\pi}{10}) + i\sin(-\frac{7\pi}{10}))$.

6. **Solution.**

(a) Suppose ζ is a complex number, and n is a positive integer. Then ζ is said to be an n-th root of unity if $\zeta^n = 1$.

(b) Let $\eta = \sin\left(\dfrac{7\pi}{30}\right) + i\cos\left(\dfrac{7\pi}{30}\right)$.

i. $\eta = \sin\left(\dfrac{7\pi}{30}\right) + i\cos\left(\dfrac{7\pi}{30}\right) = \cos\left(\dfrac{\pi}{2} - \dfrac{7\pi}{30}\right) + i\sin\left(\dfrac{\pi}{2} - \dfrac{7\pi}{30}\right) = \cos\left(\dfrac{4\pi}{15}\right) + i\sin\left(\dfrac{4\pi}{15}\right)$.

ii. We have $\eta^5 = \left(\cos\left(\frac{4\pi}{15}\right) + i\cos\left(\frac{4\pi}{15}\right)\right)^5 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \neq 1$. Hence η is not a 5-th root of unity.

We have $\eta^{10} = \left(\cos\left(\frac{4\pi}{15}\right) + i\cos\left(\frac{4\pi}{15}\right)\right)^{10} = \cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right) \neq 1$. Hence η is not a 10-th root of unity.

We have $\eta^{15} = \left(\cos\left(\frac{4\pi}{15}\right) + i\cos\left(\frac{4\pi}{15}\right)\right)^{15} = \cos(4\pi) + i\sin(4\pi) = 1$. Hence η is a 15-th root of unity.

We have $\eta^{30} = \left(\cos\left(\frac{4\pi}{15}\right) + i\cos\left(\frac{4\pi}{15}\right)\right)^{30} = \cos(8\pi) + i\sin(8\pi) = 1$. Hence η is a 30-th root of unity.

iii. The 8-th root of η with smallest positive argument is $\cos\left(\frac{\pi}{30}\right) + i\sin\left(\frac{\pi}{30}\right)$.

The 8-th roots of η with argument strictly between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$ are:—

$\cos\left(-\frac{13\pi}{60}\right) + i\sin\left(-\frac{13\pi}{60}\right)$, $\cos\left(\frac{\pi}{30}\right) + i\sin\left(\frac{\pi}{30}\right)$, $\cos\left(\frac{17\pi}{60}\right) + i\sin\left(\frac{17\pi}{60}\right)$.

7. **Solution.**

Let θ be a real number. Write $\omega = \cos(\theta) + i\sin(\theta)$.

(a) Suppose m is a positive integer.

Note that:—

- $\omega\overline{\omega} = 1$. —— ($\sharp_1$)
- $\omega + \overline{\omega} = 2\mathsf{Re}(\omega) = 2\cos(\theta)$. —— ($\sharp_2$)
- $\omega^\ell + \overline{\omega}^\ell = 2\mathsf{Re}(\omega^\ell)$ for each positive integer ℓ. —— ($\sharp_3$)

Also note that $\binom{2m}{k} = \binom{2m}{2m-k}$ for each integer k between 0 and m. —— ($\natural$)

We have

$$\begin{aligned}
& 2^{2m}\cos^{2m}(\theta) = (2\cos(\theta))^{2m} = (\omega+\overline{\omega})^{2m} \quad (\text{by } (\sharp_2)) \\
= \; & \omega^{2m} + \binom{2m}{1}\omega^{2m-2}\overline{\omega} + \binom{2m}{2}\omega^{2m-2}\overline{\omega}^2 + \cdots + \binom{2m}{k}\omega^{2m-k}\overline{\omega}^k + \cdots \\
& + \binom{2m}{m-1}\omega^{m+1}\overline{\omega}^{m-1} + \binom{2m}{m}\omega^m\overline{\omega}^m + \binom{2m}{m+1}\omega^{m-1}\overline{\omega}^{m+1} + \cdots \\
& + \binom{2m}{2m-k}\omega^k\overline{\omega}^{2m-k} + \cdots + \binom{2m}{2m-2}\omega^2\overline{\omega}^{2m-2} + \binom{2m}{2m-1}\omega\overline{\omega}^{2m-1} + \overline{\omega}^{2m} \quad (\text{by Binomial Theorem}) \\
= \; & (\omega^{2m} + \overline{\omega}^{2m}) + \binom{2m}{1}(\omega^{2m-2} + \overline{\omega}^{2m-2}) + \binom{2m}{2}(\omega^{2m-4} + \overline{\omega}^{2m-4}) + \cdots \\
& + \binom{2m}{k}(\omega^{2m-2k} + \overline{\omega}^{2m-2k}) + \cdots + \binom{2m}{m-1}(\omega^2 + \overline{\omega}^2) + \binom{2m}{m} \quad (\text{by } (\sharp_1), (\natural)) \\
= \; & 2\mathsf{Re}(\omega^{2m}) + \binom{2m}{1}\cdot 2\mathsf{Re}(\omega^{2m-2}) + \binom{2m}{2}\cdot 2\mathsf{Re}(\omega^{2m-4}) + \cdots + \binom{2m}{m-1}\cdot 2\mathsf{Re}(\omega^2) + \binom{2m}{m} \quad (\text{by } (\sharp_3))
\end{aligned}$$

(b) By De Moivre's Theorem, for each positive integer n, $\omega^n = \cos(n\theta) + i\sin(n\theta)$. Then $\mathsf{Re}(\omega^n) = \cos(n\theta)$.

By the result in the previous part,

$$\begin{aligned}
& 2^{10}\cos^{10}(\theta) \\
= \; & 2\mathsf{Re}(\omega^{10}) + \binom{10}{1}\cdot 2\mathsf{Re}(\omega^8) + \binom{10}{2}\cdot 2\mathsf{Re}(\omega^6) + \binom{10}{3}\cdot 2\mathsf{Re}(\omega^4) + \binom{10}{4}\cdot 2\mathsf{Re}(\omega^2) + \binom{10}{5} \\
= \; & 2\cos(10\theta) + 10\cdot 2\cos(8\theta) + 45\cdot 2\cos(6\theta) + 120\cdot 2\cos(4\theta) + 210\cdot 2\cos(2\theta) + 252
\end{aligned}$$

Hence

$$\cos^{10}(\theta) = \frac{1}{512}\cos(10\theta) + \frac{5}{256}\cos(8\theta) + \frac{45}{512}\cos(6\theta) + \frac{15}{64}\cos(4\theta) + \frac{105}{256}\cos(2\theta) + \frac{63}{256}$$

8. **Solution.**

(a) Suppose ζ, η are complex numbers. Write $a = \mathsf{Re}(\zeta), b = \mathsf{Im}(\zeta), c = \mathsf{Re}(\eta), d = \mathsf{Im}(\eta)$.

Note that $\zeta\eta = (a+bi)(c+di) = (ac-bd) + (ad+bc)i$.

Then $\overline{\zeta\eta} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd) - (ad+bc)i$ by the definition of complex conjugate.

We also have $\overline{\zeta} = a - bi$, $\overline{\eta} = c - di$ by the definition of complex conjugate.

Then $\overline{\zeta}\cdot\overline{\eta} = (a-bi)\cdot(c-di) = [ac + (-b)(-d)] + [a(-d) + (-b)d]i = (ac+bd) - (ad+bc)i$.

Hence $\overline{\zeta\eta} = \overline{\zeta}\cdot\overline{\eta}$.

(b) Suppose κ, λ are complex numbers.

Then by ($\natural$), we have

$$\begin{aligned} |\kappa\lambda|^2 &= (\kappa\lambda)(\overline{\kappa\lambda}) \\ &= (\kappa\lambda)(\overline{\kappa}\overline{\lambda}) \quad \text{(by } (\sharp)) \\ &= (\kappa\overline{\kappa})(\lambda\overline{\lambda}) \\ &= |\kappa|^2 \cdot |\lambda|^2 \\ &= (|\kappa| \cdot |\lambda|)^2 \quad \text{——}(\dagger) \end{aligned}$$

By definition of modulus, $|\kappa\lambda|$ is a non-negative real number.

Also by definition of modulus, $|\kappa|, |\lambda|$ are non-negative real numbers. Then $|\kappa| \cdot |\lambda|$ is a non-negative real number.

Therefore by ($\dagger$), $|\kappa\lambda| = |\kappa| \cdot |\lambda|$.

(c) Let σ, τ be complex numbers. Suppose $\tau \neq 0$.

By ($\natural\natural$), $\overline{\tau} \neq 0$ and $|\tau| \neq 0$.

We have

$$\begin{aligned} \overline{\left(\frac{\sigma}{\tau}\right)} \cdot \overline{\tau} &= \overline{\left(\frac{\sigma}{\tau} \cdot \tau\right)} \quad \text{(by } (\sharp)) \\ &= \overline{\sigma} \end{aligned}$$

Since $\overline{\tau} \neq 0$, we have $\overline{\left(\dfrac{\sigma}{\tau}\right)} = \overline{\sigma}/\overline{\tau}$.

We have

$$\begin{aligned} \left|\frac{\sigma}{\tau}\right| \cdot |\tau| &= \left|\frac{\sigma}{\tau} \cdot \tau\right| \quad \text{(by } (\flat)) \\ &= |\sigma| \end{aligned}$$

Since $|\tau| \neq 0$, we have $\left|\dfrac{\sigma}{\tau}\right| = \dfrac{|\sigma|}{|\tau|}$.

9. (a) **Answer.**

(I) $(z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w$

(II) $|z|^2 + |-w|^2 + z\overline{(-w)} + \bar{z}(-w) = |z|^2 + |w|^2 - z\bar{w} - \bar{z}w$

(III) $2|z|^2 + 2|w|^2 + z\bar{w} + \bar{z}w - z\bar{w} - \bar{z}w = 2|z|^2 + 2|w|^2$

(b) **Answer.**

(I) r, s, t are complex numbers

(II) $2|r-s|^2 + 2|r-t|^2 - |(r-s)-(r-t)|^2$

(III) $2|s-t|^2 + 2|r-s|^2 - |t-r|^2$

(IV) $|2t-r-s|^2 = 2|t-r|^2 + 2|s-t|^2 - |r-s|^2$

(V) $3(|s-t|^2 + |t-r|^2 + |r-s|^2)$

(c) **Solution.**

Let ζ, α, β be complex numbers. Suppose $\zeta^2 = \alpha^2 + \beta^2$.

By the Parallelogramic Identity, we have $|\zeta+\alpha|^2 + |\zeta-\alpha|^2 = 2|\zeta|^2 + 2|\alpha|^2$.

Then

$$\begin{aligned} (|\zeta+\alpha| + |\zeta-\alpha|)^2 &= |\zeta+\alpha|^2 + |\zeta-\alpha|^2 + 2|\zeta+\alpha||\zeta-\alpha| \\ &= 2|\zeta|^2 + 2|\alpha|^2 + 2|(\zeta+\alpha)(\zeta-\alpha)| \\ &= 2|\zeta|^2 + 2|\alpha|^2 + 2|\beta^2| \\ &= 2|\zeta|^2 + 2|\alpha|^2 + 2|\beta|^2 \end{aligned}$$

Modifying the above argument (by interchanging the roles played by α and β), we have $(|\zeta+\beta| + |\zeta-\beta|)^2 = 2|\zeta|^2 + 2|\beta|^2 + 2|\alpha|^2$.

Therefore $(|\zeta+\alpha| + |\zeta-\alpha|)^2 = (|\zeta+\beta| + |\zeta-\beta|)^2$.

Note that $|\zeta+\alpha| + |\zeta-\alpha| \geq 0$ and $|\zeta+\beta| + |\zeta-\beta| \geq 0$. Then $|\zeta+\alpha| + |\zeta-\alpha| = |\zeta+\beta| + |\zeta-\beta|$.

10. (a) **Answer.**

(I) There exists some non-zero complex number r

(II) for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$

(b) **Answer.**

(I) there exists some non-zero complex number r such that for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$

(II) $\dfrac{b_1}{b_0} = r$

(III) $\dfrac{b_m}{b_{m-1}} = r$

(IV) $\dfrac{b_1}{b_0} \cdot \dfrac{b_2}{b_1} \cdot \dfrac{b_3}{b_2} \cdot ... \cdot \dfrac{b_{m-1}}{b_{m-2}} \cdot \dfrac{b_m}{b_{m-1}} = r^m$

(V) $b_m = b_0 r^m$

(c) **Solution.**

Let $\{a_n\}_{n=0}^{\infty}$ be a geometric progression. Suppose $k, \ell, m \in \mathbb{N}$, and $a_k = A$, $a_\ell = B$ and $a_m = C$.
By the result in the previous part, there exists some non-zero complex number r such that for any $n \in \mathbb{N}$, $a_n = a_0 r^n$.
In particular, $a_k = a_0 r^k$, $a_\ell = a_0 r^\ell$ and $a_m = a_0 r^m$.
Then

$$\begin{aligned} A^{\ell-m} B^{m-k} C^{k-\ell} &= (a_0 r^k)^{\ell-m} (a_0 r^\ell)^{m-k} (a_0 r^m)^{k-\ell} \\ &= {a_0}^{(\ell-m)+(m-k)+(k-\ell)} r^{k(\ell-m)+\ell(m-k)+m(k-\ell)} \\ &= {a_0}^0 r^0 = 1 \end{aligned}$$

11. (a) **Answer.**

(I) Suppose a, b, c are in arithmetic progression.
(II) d
(III) $c - b = d$
(IV) $b^2 - a^2 - ca + bc = (b-a)(b+a) + (b-a)c = d(a+b+c)$
(V) $[(c^2 - ab) - (b^2 - ca)] = c^2 - b^2 - ab + ca = (c-b)(c+b) + (c-b)a = d(a+b+c)$
(VI) $(b^2 - ca) - (a^2 - bc) = (c^2 - ab) - (b^2 - ca)$

(b) **Solution.**

Let a, b, c be complex numbers. Suppose $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression. Further suppose $a + b + c \neq 0$.

By assumption, $b^2 - ca = \dfrac{(a^2 - bc) + (c^2 - ab)}{2}$.
Therefore $2b^2 - 2ca = a^2 + c^2 - ab - bc$.
Hence $0 = (a^2 - b^2) + (c^2 - b^2) + (ac - ab) + (ac - bc) = \cdots = (a + c - 2b)(a + b + c)$.

Since $a + b + c \neq 0$, we have $a + c - 2b = 0$. Then $b = \dfrac{a+b}{2}$.

Therefore a, b, c are in arithmetic progression.