MATH1050BC/1058 Assignment 1

Advice.

1. The questions in this assignment are mainly about the 'algebra' of complex numbers. Do familiarize yourself with the corresponding material available in the course homepage before trying the questions.
2. (a) Questions (1)-(7) are more about manipulations, computations and pictorial visualizations.
   (b) Questions (8)-(11) are more about proof-writing.
3. Questions which require more thought and/or work and/or tricks and/or organization and/or ... than the 'unlabelled' questions are labelled by $\diamondsuit, \clubsuit, \heartsuit, \spadesuit$ in ascending order of overall difficulty level.

Instructions.

1. Any work submitted by you must be written on **A4-size sheets** and must be appropriately binded.

   Your name and student ID, as in your student card, and the **code** of the section to which you are registered must be written at the upper right corner of the first page of your submission.

2. (a) **Mandatory work, for assessment purpose.**

   You are **required** to submit work on Questions (1)-(7), (9a), (9b), (10a), (10b), (11a), for course assessment purpose.

   (b) **Optional proof-writing exercise.**

   You may also **opt** to submit, **on exactly one sheet, separate from your submission on mandatory work**, your work on Questions (8a), (8b). It will be read and commented, but not counted for course assessment.

   (c) **Other optional work.**

   You may choose submit work on other questions in this assignment not mentioned above, alongside the mandatory work, but there is no guarantee that it will be read.

3. You must adhere to the notations which have been in the course. Things which have not been formally defined in the course are not allowed in your work.

   (So, you are not allowed to use the short-hand symbol 'cis', and not allowed to write the 'e^z', '$\exp(z)$', '$\cos(z)$', '$\sin(z)$' whenever z is not a real number. You are not allowed to write '$\sqrt[n]{z}$' whenever z is not a non-negative real number.)

* * *

1. Let $\omega = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$.

   (a) i. Express ω in polar form, displaying an argument which is between $-\pi$ and π.

   ii. Write down the respective values of ω^2, ω^3, ω^5, ω^6, ω^{11}, ω^{12}. Give your answers in standard form.

   (b)$^\diamondsuit$ Hence, or otherwise, find the value of $\displaystyle\sum_{k=0}^{2230} \omega^{k+1050}$. Justify your answer.

   Remarks.

   (1) You should observe that the sum in question is the sum of finitely many consecutive terms of a geometric progression.

   (2) At some point you may want to write down equalities that read '$\omega^{1050} = \omega^m$', '$\omega^{2231} = \omega^n$' in which m, n are some appropriate integers between 0 and 11. You may need 'index laws' about (integral) powers.

2. Let α, β, γ be complex numbers. Suppose $|\alpha| = 3$, $|\beta| = \dfrac{1}{4}$ and $|\gamma| = 5$.

   (a) What is the value of $\left|\dfrac{i \cdot \alpha^3 \cdot \overline{\beta} \cdot \overline{\gamma^3}}{\overline{\alpha} \cdot \beta^2 \cdot \gamma^2}\right|$? Justify your answer.

   (b) It is further supposed that $\mathsf{Re}(\alpha) = 2$, $\mathsf{Im}(\dfrac{1}{\beta^2}) = 14$, and $\mathsf{Re}(i\gamma^3) = 100$.

   i. What is the value of $|\mathsf{Im}(\alpha)|$? Justify your answer.

   ii. What is the value of $\left|\mathsf{Re}(\dfrac{1}{\beta^2})\right|$? Justify your answer.

   iii.$^\diamondsuit$ What is the value of $\mathsf{Im}(\dfrac{\overline{\gamma}^2}{\gamma})$? Justify your answer.

3. (a) Explain the phrase '*square root of a complex number*' by providing an appropriate definition.

   (b) Let r be a positive real number, and θ be a real number. Suppose $\zeta = r(\cos(\theta) + i\sin(\theta))$.
   How many square roots are there for the number ζ? Write down all of them explicitly. Give your answer in terms of r and θ. You are not required to justify your answers.

   (c) Let α be a non-zero complex number. Suppose $\alpha^5/\overline{\alpha}^3 = 4$. Also suppose that α is neither real nor purely imaginary.

      i. What is the modulus of α? Justify your answer.
      Remark. Make good use of equalities about modulus alone, or about modulus and conjugate. You do not not need those which explicitly involve real part, imaginary part or argument.

      ii.$^\diamond$ Find all possible values of α. Express your answers in standard form. Justify your answer.
      Remark. At some point you need to think of what you mean by 'square root(s) of the complex number so-and-so'. Do not be tempted to write the symbol '$\sqrt{\zeta}$' when ζ is not a non-negative real number, because such a symbol makes no sense.

4. *The three parts of this question are un-related. There is no need to give any justifications for your answers in this question. For each part, always ask: What are the curves described by the respective equations in the Argand plane?*

   (a) Write down all solutions of the system of equations $\begin{cases} |z-2-2i| &= 2 \\ |z-4+2i| &= |z-2i| \end{cases}$ with complex unknown z.

   (b)$^\diamond$ Consider the system of equations $(S_{k,\ell,a}) : \begin{cases} |z-2-3i| &= k \\ |z-a-3i| &= \ell \end{cases}$ with complex unknown z. Here a is some real number and k, ℓ are some non-negative real number.
   Suppose $(S_{k,\ell,a})$ has at most two solutions, and $5+7i$ is a solution of $(S_{k,\ell,a})$.

      i. What is the value of k?
      ii. Name every solution of $(S_{k,\ell,a})$ other than $5+7i$, if such exists.

   Remark. There is no need to find the values of ℓ and a.

   (c)$^\diamond$ Consider the system of equations $(S_{\alpha,r}) : \begin{cases} |z-2i| &= 2 \\ |z-4-4i| &= |z| \\ |z-\alpha| &= r \end{cases}$ with complex unknown z. Here α is some complex number and r is a non-negative real number.
   Suppose that $(S_{\alpha,r})$ has two distinct solutions.

      i. Write down all solutions of $(S_{\alpha,r})$.
      ii. What is the smallest possible value of r?
      iii. What is the value of α if $|\mathsf{Re}(\alpha)| = |\mathsf{Im}(\alpha)|$?

5. (a) Explain the phrase '*n-th root of a complex number*' by giving an appropriate definition.

   (b) *There is no need to give any justifications for your answers in this question.*

      i. Write down all the quintic roots of $32i$. Leave your answer in polar form with argument between $-\pi$ and π.
      ii.$^\diamond$ Hence, or otherwise, write down all the solutions of the system of inequalities

$$\begin{cases} z^5 - 32i &= 0 \\ \mathsf{Re}(z) &\geq \mathsf{Im}(z) \end{cases} \quad \text{with complex unknown } z.$$

   Remark. Drawing some appropriate picture(s) on the Argand plane will help.

6. (a) Explain the phrase '*root of unity*' by providing an appropriate definition.

   (b) Let $\eta = \sin\left(\dfrac{7\pi}{30}\right) + i\cos\left(\dfrac{7\pi}{30}\right)$.

      i. Express η in polar form, with an argument between $-\pi$ and π
      ii. Is η a 5-th root of unity? Is η a 10-th root of unity? Is η a 15-th root of unity? Is η a 30-th root of unity? Justify your respective answers with reference to the definition for root of unity.
      iii.$^\diamond$ Write down the 8-th root of η with the smallest positive argument. Hence, or otherwise, write down all 8-th roots of η with argument strictly between $-\dfrac{\pi}{3}$ and $\dfrac{\pi}{3}$. Leave your answers in polar form.

   Remark. Drawing some appropriate picture(s) on the Argand plane will help.

7. Let θ be a real number. Write $\omega = \cos(\theta) + i\sin(\theta)$.

(a)♣ Suppose m is a positive integer.

By applying Binomial Theorem, or otherwise, deduce that

$$2^{2m}\cos^{2m}(\theta) = 2\mathsf{Re}(\omega^{2m}) + \binom{2m}{1}\cdot 2\mathsf{Re}(\omega^{2m-2}) + \binom{2m}{2}\cdot 2\mathsf{Re}(\omega^{2m-4}) + \cdots + \binom{2m}{m-1}\cdot 2\mathsf{Re}(\omega^{2}) + \binom{2m}{m}$$

(b) Hence, or otherwise, show that the equality holds

$$\cos^{10}(\theta) = a_5\cos(10\theta) + a_4\cos(8\theta) + a_3\cos(6\theta) + a_2\cos(4\theta) + a_1\cos(2\theta) + a_0$$

for some appropriate choice of positive rational numbers $a_0, a_1, a_2, a_3, a_4, a_5$.

Remark. Combining such an 'identity' on θ with the Taylor series expansion of cos about 0, we can express $\cos^{10}(\theta)$ as an infinite series of ascending non-negative integral powers of θ with relative ease.

8. (a) Prove the statement ($\sharp$) below, with reference to the definitions of real part, imaginary part and complex conjugate:—

($\sharp$) *Suppose ζ, η are complex numbers. Then $\overline{\zeta\eta} = \overline{\zeta}\cdot\overline{\eta}$.*

(b) Take for granted the validity of the statement ($\natural$):—

($\natural$) *Suppose ζ is a complex number. Then $\zeta\cdot\overline{\zeta} = |\zeta|^2$.*

By applying ($\sharp$) and ($\natural$), or otherwise, deduce the statement ($\flat$):—

($\flat$) *Suppose κ, λ are complex numbers. Then $|\kappa\lambda| = |\kappa|\cdot|\lambda|$.*

Remarks.

(1) You are not required to go into the level of rigour in making explicit reference to the respective Laws of Commutativity and Laws Associativity for the reals or complex numbers.

(2) At some point, you may need to apply these *beliefs* about the real number system (which can be rigorously established as theorems upon some appropriately formulated axioms about inequalities for the reals):—

($*$) *Suppose s, t are non-negative real numbers. Then st is a non-negative real number.*

($**$) *Suppose s, t are non-negative real numbers. Further suppose $s^2 = t^2$. Then $s = t$.*

(c) Take for granted the validity of the statement ($\natural\natural$):—

($\natural\natural$) *Suppose ζ is a non-zero complex number. Then $\overline{\zeta} \neq 0$ and $|\zeta| \neq 0$.*

By applying ($\sharp$), ($\flat$), ($\natural\natural$), or otherwise, prove the statement ($\boxtimes$):—

($\boxtimes$) *Let σ, τ be complex numbers. Suppose $\tau \neq 0$. Then $\overline{\left(\frac{\sigma}{\tau}\right)} = \overline{\sigma}/\overline{\tau}$ and $\left|\frac{\sigma}{\tau}\right| = \frac{|\sigma|}{|\tau|}$.*

9. (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A), which is known as the **Parallelogramic Identity**. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(A) *Suppose z, w are complex numbers. Then $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.*

> Suppose z, w are complex numbers.
> Then $|z+w|^2 = (z+w)\overline{(z+w)} =$ ______(I)______ .
> Also, $|z-w|^2 = |z+(-w)|^2 =$ ______(II)______ .
> Then $|z+w|^2 + |z-w|^2 =$ ______(III)______ .

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(B) *Suppose r, s, t are complex numbers. Then $|2r-s-t|^2+|2s-t-r|^2+|2t-r-s|^2 = 3(|s-t|^2+|t-r|^2+|r-s|^2)$.*

> Suppose ______(I)______ .
> Then, by the Parallelogramic Identity,
>
> $|2r-s-t|^2 = |(r-s)+(r-t)|^2 =$ ______(II)______ $= 2|r-s|^2 + 2|t-r|^2 - |s-t|^2$.
>
> Similarly, $|2s-t-r|^2 =$ ______(III)______ .
> Also, ______(IV)______ .
> Therefore $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 =$ ______(V)______ .

(c)$^\diamond$ Applying the Parallelogramic Identity, or otherwise, prove the statement (†) below.

(†) *Let ζ, α, β be complex numbers. Suppose $\zeta^2 = \alpha^2 + \beta^2$. Then $|\zeta + \alpha| + |\zeta - \alpha| = |\zeta + \beta| + |\zeta - \beta|$.*

Remark. At some point you may want to make good use of the 'formula' $|\kappa\lambda| = |\kappa| \cdot |\lambda|$ which holds for any complex numbers κ, λ.

10. (a) Fill in the blanks in the passage below so as to give the definition for the notion of *geometric progression*:

Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero complex numbers.

The infinite sequence $\{b_n\}_{n=0}^{\infty}$ is said to be a **geometric progression** *if the statement (GP) holds:*

(GP) ______(I)______ such that ______(II)______ .

The number r is called the **common ratio** *of this geometric progression.*

(b) Consider the statement (E):

(E) *Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero complex numbers. Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression. Then there exists some non-zero complex number r such that for any $m \in \mathbb{N}$, $b_m = b_0 r^m$.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (E). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let $\{b_n\}_{n=0}^{\infty}$ is an infinite sequence of non-zero complex numbers. Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression.
>
> Then ______(I)______ .
>
> Pick any $m \in \mathbb{N}$. By the definition of geometric progression, the equalities
>
> $$\underline{\qquad (II) \qquad}, \ \frac{b_2}{b_1} = r, \ \frac{b_3}{b_2} = r, \ \ldots, \ \frac{b_{m-1}}{b_{m-2}} = r, \text{ and } \underline{\qquad (III) \qquad}$$
>
> hold simultaneously.
>
> Then $\dfrac{b_m}{b_0} =$ ______(IV)______.
>
> Therefore ____(V)____ .

(c) Hence, or otherwise, prove the statement (♯):

(♯) *Let $\{a_n\}_{n=0}^{\infty}$ be a geometric progression. Suppose $k, \ell, m \in \mathbb{N}$, and $a_k = A$, $a_\ell = B$ and $a_m = C$. Then $A^{\ell-m} B^{m-k} C^{k-\ell} = 1$.*

11. (a) Consider the statement (F):

(F) *Let a, b, c be complex numbers. Suppose a, b, c are in arithmetic progression. Then $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (F). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let a, b, c be complex numbers. ______(I)______
>
> Denote the common difference by __(II)__ .
>
> [We want to verify that $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.
> By definition, it suffices to verify that $(b^2 - ca) - (a^2 - bc) = (c^2 - ab) - (b^2 - ca)$.]
>
> By definition, we have $b - a = d$ and ____(III)____ .
>
> Note that $(b^2 - ca) - (a^2 - bc) =$ ______(IV)______.
>
> Also note that ______(V)______ .
>
> Then ______(VI)______ .
>
> Hence $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.

(b)$^\diamond$ Prove the statement (♮):

(♮) *Let a, b, c be complex numbers. Suppose $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression. Further suppose $a + b + c \neq 0$. Then a, b, c are in arithmetic progression.*

Remark. You may take for granted the result (♮):

(♮) *Suppose u, v, w are numbers. Then u, v, w are in arithmetic progression iff $v = \dfrac{u+w}{2}$.*