# MATH 1510 Chapter 8

## 8.1 Power Series

Roughly speaking, a power series is a polynomial with degree $\infty$.

**Definition 8.1** (Power series)**.** A **power series** is a function of the form

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} c_k (x-a)^k \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned}$$

where $a, c_0, c_1, c_2, \dots,$ are real numbers. We call $a$ the **center** of the series.

The implied domain of a power series is the set of $x$ such that it converges.

It is known that any power series either converges everywhere, or there is a number $R$ such that $f(x)$ converges for all $x \in (a-R, a+R)$ and diverges for all $x$ such that $|x-a| > R$.

This number $R$ is called the **radius of convergence** of the series. For convenience, if $f(x)$ converges everywhere, we say that its radius of convergence is $R = \infty$.

It is also known that

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right|$$

if the limit exists or is equal to infinity.

(If the limit does not exist, it does not mean $R$ does not exist. It just means that one would have to use some other method to find $R$.)

**Example 8.2.** Consider the power series

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots .$$

Clearly, $a = 0$, $c_k = 1$ and so,

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = 1 \implies R = 1.$$

Since

$$f\left(\frac{1}{2}\right) = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \text{ converges(geometric series)}$$
$$f(2) = 1 + 2 + 2^2 + 2^3 + \cdots \text{ diverges,}$$

we know that $\frac{1}{2} \in D_f$ but $2 \notin D_f$.

**Proposition 8.3** (Interval of convergence)**.** *The implied domain of any power series $f(x)$ with center $a$ and radius of convergence $R$ is an interval of the form:*

$$(a-R, a+R), (a-R, a+R], [a-R, a+R) \text{ or } [a-R, a+R].$$

*Proof of Interval of convergence.* Let us handle the case when $R \neq 0, \infty$. The cases when $R = 0$ and $R = \infty$ follow from similar arguments. Suppose $|x - a| < R$. Then, for the series $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$,

$$\lim_{k\to\infty} \left| \frac{(k+1)\text{-th term}}{k\text{-th term}} \right| = \lim_{k\to\infty} \left| \frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k} \right| = \frac{|x-a|}{R} < 1$$

Therefore, by ratio test, the series converges.On the other hand, if $|x - a| > R$, then

$$\lim_{k\to\infty} \left| \frac{(k+1)\text{-th term}}{k\text{-th term}} \right| = \lim_{k\to\infty} \left| \frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k} \right| = \frac{|x-a|}{R} > 1$$

Again, by ratio test, the series diverges.Hence,

$$(a-R, a+R) \subseteq D_f \subseteq [a-R, a+R]$$

and the result follows. □

Thus, we call the implied domain of a power series its **interval of convergence**.

**Example 8.4.** As in Example Example 8.2,

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots .$$

Since its center and radius of convergence are $0, 1$ respectively, we can conclude that the interval of convergence of $f(x)$ is either

$$(-1, 1), (-1, 1], [-1, 1) \text{ or } [-1, 1].$$

That means $f(x)$ converges whenever $x \in (-1, 1)$. In fact, for any $x \in (-1, 1)$,

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

**Example 8.5.** For the following power series, find its center and radius of convergence. For what $x$ does the series converge?

- $$f(x) = \sum_{k=0}^{\infty} (k!)(x-1)^k$$

- $$f(x) = \sum_{k=0}^{\infty} (-1)^k \left(\sin 2^{-k}\right) x^k$$

## 8.2 Taylor Series

While a calculator can only perform basic arithmetic: $+, -, \times, \div$, how does it compute something like $\sin 1$ or $e^{\pi}$ ? The answer is **Taylor series** .

**Definition 8.6** (Taylor series)**.** We say that a function $f(x)$ is **smooth** (or **infinitely differentiable**) over an interval $I$ if $f^{(n)}(x)$ is differentiable over $I$ for any $n \geq 0$. The Taylor series of a smooth function $f(x)$ at a point $x = a$ is:

$$\begin{aligned} T(x) &= \sum_{k=0}^{\infty} c_k (x-a)^k \quad \text{where } c_k = \frac{f^{(k)}(a)}{k!} \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

The **Maclaurin series** of $f(x)$ is its Taylor series with center $a = 0$ :

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots .$$

The **$n$-th Taylor Polynomial** (or **Taylor polynomial of order** $n$) of $f(x)$ at a point $x = a$ is:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Expanded up to order $n$ ) The **$n$-th Maclaurin Polynomial** (or **Maclaurin polynomial of order** $n$) of $f(x)$ is its Taylor polynomial of order $n$ with center $a = 0$ :

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n.$$

**Remark.** Observe that the Taylor polynomial $T_n(x)$ of $f(x)$ at $x = a$ is the unique polynomial which satisfies the condition:

$$T_n^{(k)}(a) = f^{(k)}(a), \quad 0 \le k \le n.$$

**Example 8.7.** Consider the function $f(x) = e^x$. It's clearly smooth over $\mathbb{R}$. Moreover,

$$f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1 \text{ and } f^{(n)}(1) = e.$$

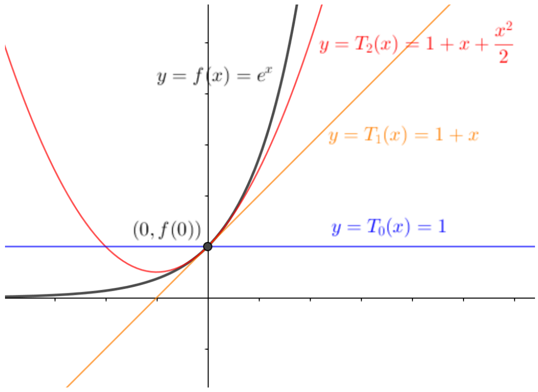
Therefore,

$$\text{Maclaurin series of } f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \cdots$$

Taylor series of $f(x)$ about $x = 1$ is $\displaystyle\sum_{k=0}^{\infty} \frac{e}{k!}(x-1)^k = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots$ .

By definition, the Maclaurin polynomials of $f(x)$ of orders $0, 1, 2$ are

$$1, 1 + x, 1 + x + \frac{1}{2} x^2 \text{respectively.}$$




From Example Example 8.7, we can see that Taylor polynomial of order $n$ can be regarded as a degree $n$ polynomial approximation of $f$ around the center $a$. In particular,

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the linearization of $f$ at $a$.

- Taylor polynomials of $f(x) = \sin x$ centered at $a = 0$.
- Taylor polynomials of $f(x) = \sin x$ centered at $a = \pi/2$.

The following are some basic Taylor series:

**Proposition 8.8.**

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

*converges for all* $x \in \mathbb{R}$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

*converges for all* $x \in \mathbb{R}$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

*converges for all* $x \in \mathbb{R}$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

*converges for all* $x \in (-1, 1)$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

*converges for all* $x \in (-1, 1]$

**Remark.** In general, a function and its Taylor series are not necessarily equal to each other as functions.

For instance, the domain of the Maclaurin series of $\frac{1}{1-x}$ is $(-1, 1)$, while the domain of $\frac{1}{1-x}$ is $(-\infty, 1) \cup (1, \infty)$.

## 8.3 Operations on Taylor Series

It is known that if a power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ converges to a given function $f(x)$ on an open interval centered at $x = a$, then that power series *is* the Taylor series of $f(x)$ at $x = a$:

$$c_k = \frac{f^{(k)}(a)}{k!}$$

This implies in particular that the power series centered at $x = a$ converging to the function $f(x)$ on an open interval is *unique* : There cannot be another power series with the same center which also converges to the same function on an open interval.

This fact offers a "shortcut" to find the Taylor series of various functions based on known Taylor series.

Suppose

$$f(x) = \sin x, \quad \text{and } g(x) = \cos x.$$

(For the following Taylor series, the centers are assumed to be $a = 0$.)

$$\begin{aligned}\text{Taylor series of } (f(x) + g(x)) &= \text{Taylor series of } f(x) + \text{Taylor series of } g(x) \\ &= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\end{aligned}$$

$$\text{Taylor series of } (f(x) - g(x)) = \text{Taylor series of } f(x) - \text{Taylor series of } g(x)$$
$$= -1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

$$\text{Taylor series of } (f(x) \cdot g(x)) = (\text{Taylor series of } f(x)) \cdot (\text{Taylor series of } g(x))$$
$$= x - \frac{2x^3}{3} + \cdots$$

$$\text{Taylor series of } g(f(x)) = \text{Taylor series of } g(y) \text{ with } y = \text{Taylor series of } f(x)$$
$$= 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots$$

$$\text{Taylor series of } f'(x) = \text{Differentiating Taylor series of } f(x) \text{ term by term}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(Notice that this coincides with the Taylor series of $g(x) = \cos x$.)

$$\text{Taylor series of } \int_0^x f(t)\,dt = \text{Integrating Taylor series of } f(t) \text{ term by term}$$
$$= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$$

(Notice that this coincides with the Taylor series of $1 - \cos x$.)

To find the Taylor series of $\frac{f(x)}{g(x)}$ where $g(a) \neq 0$, we start by letting:

$$\text{Taylor series of } \frac{f(x)}{g(x)} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Then,

$$\text{Taylor series of } f(x) = \left(\text{Taylor series of } \frac{f(x)}{g(x)}\right) \cdot (\text{Taylor series of } g(x))$$

$$x - \frac{1}{6}x^3 + \cdots$$
$$= (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right)$$

Hence, by comparing the coefficients, we have

$$\begin{array}{lrcll} x^0 \text{ term:} & 0 & = & c_0(1) & \implies c_0 = 0 \\ x^1 \text{ term:} & 1 & = & c_0(0) + c_1(1) & \implies c_1 = 1 \\ x^2 \text{ term:} & 0 & = & c_0\left(-\frac{1}{2}\right) + c_1(0) + c_2(1) & \implies c_2 = 0 \\ x^3 \text{ term:} & -\frac{1}{6} & = & c_0(0) + c_1\left(-\frac{1}{2}\right) + c_2(0) + c_3(1) & \implies c_3 = \frac{1}{3} \end{array}$$

$$x^4 \text{ term:} \quad 0 = c_0\left(\frac{1}{24}\right) + c_1(0) + c_2\left(-\frac{1}{2}\right) + c_3(0) + c_4(1)$$

$$\implies c_4 = 0$$

and we can conclude that:

$$\text{Taylor series of } \frac{f(x)}{g(x)} = x + \frac{1}{3}x^3 + \cdots$$

**Example 8.9.** • Find the Maclaurin series of $f(x) = \sin^2 x$.

• Hence, find $f^{(10)}(0)$ and $f^{(11)}(0)$.

**Example 8.10.** Find the Maclaurin series of $f(x) = \sqrt{1 + x^2}$.

**Example 8.11.** Find the Maclaurin series of $f(x) = \dfrac{x}{1 - x^3}$.

**Example 8.12.** Find the Maclaurin series of $f(x) = \arctan x$.

**Example 8.13.** Find the Taylor series of $f(x) = \dfrac{x}{x+1}$ with center $a = 1$.

**Example 8.14.** Find the Maclaurin polynomial of order 3 of $f(x) = e^{\cos x}$.

## 8.4 Lagrange Form of Remainder

Although Taylor series is powerful, no machine can really perform an infinite sum. So in practice, a calculator computes a finite sum with acceptable error instead. That means we need to control the error.

**Theorem 8.15** (Taylor's Theorem)**.** *Suppose $f(x)$ is $(n+1)$-times differentiable over the interval $[a, x]$ (or $[x, a]$). Let $T_n(x)$ be the $n$-th Taylor polynomial of $f(x)$ centered at $x = a$. Then:*

$$f(x) = T_n(x) + R_n(x),$$

*where:*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

*for some* $c \in (a,x)$ *(or* $(x,a)$ *).* $R_n(x)$ *is called the* **Lagrange form of the remainder**.

**Remark.** Be careful: $R_n(x)$ looks similar to the $(x-a)^{n+1}$ term of the Taylor series, but is not the same.

*Proof of Taylor's Theorem.* Let:

$$F(t) = f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n$$

and $G(t) = (t-x)^{n+1}$. Then $F(t)$ is differentiable over $[a,x]$ (or $[x,a]$ ) and $G'(t) \neq 0$ over $(a,x)$ (or $(x,a)$ ). Notice that $F(x) = f(x)$, $F(a) = T_n(x)$ and

$$\begin{aligned} F'(t) &= \frac{d}{dt}\left(f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n\right) \\ &= f'(t) + (f''(t)(x-t) - f'(t)) + \frac{1}{2!}(f'''(t)(x-t)^2 - 2f''(t)(x-t)) \\ &\quad + \cdots + \frac{1}{n!}(f^{(n+1)}(t)(x-t)^n - nf^{(n)}(t)(x-t)^{n-1}) \\ &= \frac{1}{n!}f^{(n+1)}(t)(x-t)^n, \end{aligned}$$

Therefore, by Theorem 5.9 (Cauchy's Mean Value Theorem), there exists $c \in (a,x)$ (or $(x,a)$ ) such that

$$\begin{aligned} \frac{F'(c)}{G'(c)} &= \frac{F(x)-F(a)}{G(x)-G(a)} \\ \frac{\frac{1}{n!}f^{(n+1)}(c)(x-c)^n}{(n+1)(c-x)^n} &= \frac{f(x)-T_n(x)}{-(a-x)^{n+1}} \\ \frac{f^{(n+1)}(c)}{(n+1)!}(-1)^n(-1)(a-x)^{n+1} &= f(x)-T_n(x) \end{aligned}$$

Hence, $f(x) = T_n(x) + R_n(x)$ as desired. □

Alternatively,

*Proof of Taylor's Theorem.* Recall that $T_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \ldots, n$. Moreover, observe that $T_n^{(k)} = 0$ for $k > n$, since $T_n$ is a polynomial of degree at most $n$.

Let:

$$F(x) = f(x) - T_n(x), \quad G(x) = (x-a)^{n+1}.$$

Then, $F(a) = G(a) = 0$, and by Theorem 5.9 (Cauchy's Mean Value Theorem), we have:

$$\begin{aligned}
\frac{f(x) - T_n(x)}{(x-a)^{n+1}} &= \frac{F(x) - F(a)}{G(x) - G(a)} \\
&= \frac{F'(x_1)}{G'(x_1)} \\
&= \frac{f'(x_1) - T_n'(x_1)}{(n+1)(x_1 - a)^n}
\end{aligned}$$

for some $x_1$ between $a$ and $x$.

Now let:

$$\begin{aligned}
F_1(x) = F'(x) = f'(x) - T_n'(x), \\
G_1(x) = G'(x) = (n+1)(x-a)^n.
\end{aligned}$$

Repeating the same procedure carried out before, we have:

$$\frac{f'(x_1) - T_n'(x_1)}{(n+1)(x_1 - a)^n} = \frac{F_1'(x)}{G_1'(x)} = \frac{f^{(2)}(x_2) - T_n^{(2)}(x_2)}{(n+1)n(x_2 - a)^{n-1}}$$

for some $x_2$ between $a$ and $x_1$. Repeating this process $n+1$ times, we have:

$$\begin{aligned}
\frac{f(x) - T_n(x)}{(x-a)^{n+1}} &= \frac{f'(x_1) - T_n'(x_1)}{(n+1)(x_1 - a)^n} \\
&= \frac{f^{(2)}(x_2) - T_n^{(2)}(x_2)}{(n+1)n(x_2 - a)^{n-1}} \\
&\vdots \\
&= \frac{f^{(n)}(x_n) - T_n^{(n)}(x_n)}{(n+1)n(n-1)\cdots 2(x_n - a)} \\
&= \frac{f^{(n+1)}(x_{n+1}) - 0}{(n+1)!}
\end{aligned}$$

for some $x_{n+1}$ between $a$ and $x$. Letting $c = x_{n+1}$, the theorem follows. □

**Remark.** If we apply the Taylor's Theorem with $n = 0$, we have

$$f(x) = T_0(x) + R_0(x) = f(a) + f'(c)(x - a)$$

$$\Longrightarrow \frac{f(x) - f(a)}{x - a} = f'(c)$$

Thus, Taylor's Theorem can be regarded as a generalization of Lagrange's MVT.

**Example 8.16.** For any $x > 0$,

$$e^x = T_3(x) + R_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{e^c}{4!}x^4$$

for some $c \in (0, x)$. In particular, when $x = 1$,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{e^c}{24}$$

with $c \approx 0.214114 \in (0, 1)$.

**Example 8.17.** To approximate the value of $\sin 1$, we apply Taylor's Theorem on $\sin x$ :

$$\sin x = T_4(x) + R_4(x) = x - \frac{1}{3!}x^3 + \frac{\cos c}{5!}x^5$$

where $c \in (0, x)$. By putting $x = 1$, we have:

$$\frac{5}{6} - \frac{1}{120} \leq \sin 1 = \frac{5}{6} + \frac{\cos c}{120} \leq \frac{5}{6} + \frac{1}{120}$$

$$0.825 \leq \sin 1 \leq 0.8416667$$

(In fact, $\sin 1 \approx 0.841471$ )

**Example 8.18.** Let's try to approximate

$$\int_0^1 \cos(x^2)\, dx$$

with an error $< 0.001$. First of all, we apply Taylor's Theorem on $\cos t$ :

$$\begin{aligned}\cos t &= T_n(t) + R_n(t), \qquad \text{where } n = 2m \text{ is even,}\\ &= 1 - \frac{1}{2!}t^2 + \cdots + (-1)^m \frac{1}{(2m)!}t^{2m} + \frac{(-1)^{m+1}\sin c}{(2m+1)!}t^{2m+1}\end{aligned}$$

for some $c \in (0, t)$. By putting $t = x^2$, we have

$$\begin{aligned}
\text{Exact value } &= \int_0^1 \cos(x^2)\,dx \\
&= \int_0^1 \left(1 - \frac{1}{2!}x^4 + \cdots + (-1)^m \frac{1}{(2m)!}x^{4m}\right) dx \\
&\quad + \int_0^1 (-1)^{m+1} \frac{\sin c}{(2m+1)!} x^{4m+2}\,dx \\
&= \text{Approximation} + \text{Error}
\end{aligned}$$

So, we can see that:

$$\begin{aligned}
|\text{Error}| &= \left| \int_0^1 (-1)^{m+1} \frac{\sin c}{(2m+1)!} x^{4m+2}\,dx \right| \le \int_0^1 \frac{1}{(2m+1)!} x^{4m+2}\,dx \\
&= \frac{1}{(2m+1)!(4m+3)},
\end{aligned}$$

which would be $< 0.001$ when $m = 2$. Hence, with $m = 2$,

$$\text{Approximation} = \int_0^1 \left(1 - \frac{1}{2!}x^4 + \frac{1}{4!}x^8\right) dx \approx 0.9046296.$$

(In fact, $\displaystyle\int_0^1 \cos(x^2)\,dx \approx 0.9045242$.)

**Example 8.19.** Find the exact value of

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$