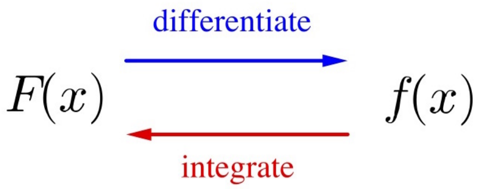
MATH 1510 Chapter 6

6.1 Indefinite integral

Integration is nothing but the reverse of differentiation.



To be more precise,

Definition 6.1. We call $F(x)$ an **antiderivative** of $f(x)$ if:

$$\frac{d}{dx}F(x) = f(x).$$

The collection of all antiderivatives of $f(x)$ is denoted by:

$$\int f(x)\,dx,$$

also called the **indefinite integral** of $f(x)$.

(For now, "dx " would just be part of the notation.)

Example 6.2. Since:

$$\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x,$$

the function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$.

Notice that we also have:

$$\frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{d}{dx}\left(\frac{1}{2}x^2 + 1\right) = \frac{d}{dx}\left(\frac{1}{2}x^2 - \pi\right) = x.$$

Hence, the expressions:

$$\frac{1}{2}x^2, \quad \frac{1}{2}x^2 + 1, \quad \frac{1}{2}x^2 - \pi$$

all give antiderivatives of $f(x) = x$.

In fact, a function is an antiderivative of $f(x) = x$ if and only if it is equal to $\frac{1}{2}x^2 + C$ for some constant $C \in \mathbb{R}$.

Hence, we may represent the collection of all antiderivatives of $f(x) = x$ as follows:

$$\int x\,dx = \frac{1}{2}x^2 + C,$$

where C is an arbitrary constant.

Proposition 6.3. *For any constants $a, b, k \in \mathbb{R}$,*

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx;$$

$$\int x^k\,dx = \frac{1}{k+1}x^{k+1} + C \quad \text{if } k \neq -1;$$

$$\int x^{-1}\,dx = \ln|x| + C;$$

$$\int \sin x\,dx = -\cos x + C;$$

$$\int \cos x\,dx = \sin x + C;$$

$$\int e^x\,dx = e^x + C;$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C;$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C;$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

Proof of Proposition 6.3. When $x > 0$, we have

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

On the other hand, if $x < 0$, we have

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

as desired. The other identities are just direct consequences of differentiation. $\square$

Example 6.4.

- $$\int \left(\cos x + \frac{2}{x} - 3^x\right) dx;$$

- $$\int \frac{x^4}{1+x^2}\, dx.$$

6.2 Integration by Substitution

Sometimes, integration can be handled by "change of variable":

Theorem 6.5 (Integration by Substitution)**.** *Assuming differentiability and integrability, suppose*

$$y = g(x) \quad \textit{and} \quad f(x) = h(y)\frac{dy}{dx}.$$

Then,

$$\int f(x)\, dx = \int h(y)\, dy$$

Proof of Integration by Substitution. Let $H(y)$ be the antiderivative of $h(y)$. Then,

$$\frac{d}{dx}H(g(x)) = H'(g(x))g'(x) = h(g(x))g'(x) = h(y)\frac{dy}{dx} = f(x)$$

Hence,

$$\int f(x)\, dx = H(g(x)) + C = H(y) + C = \int h(y)\, dy$$

$\square$

The formula is easier to remember if we use notations like $\left(\frac{dy}{dx}\right) dx = dy$, in which the part "$dx$ " becomes crucial.

Example 6.6. To evaluate

$$\int e^x \sin(e^x)\, dx,$$

we let $u = e^x$. By the fact that

$$\frac{du}{dx} = e^x \implies du = e^x\, dx$$

we have

$$\begin{aligned}\int e^x \sin(e^x)\, dx &= \int \sin(e^x)(e^x\, dx) \\ &= \int \sin u\, du \\ &= -\cos u + C \\ &= -\cos(e^x) + C.\end{aligned}$$

Remember to change everything into u.

Example 6.7.

- Evaluate

$$\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}}\, dx$$

by the substition $u = e^{\sqrt{x}}$.

- Evaluate

$$\int x \sin(x^2)\, dx.$$

Example 6.8. Sometimes, if it's not too complicated, it might be preferable to do the substitution without introducing a new variable:

$$\begin{aligned}\int \frac{1}{x-2}\, dx &= \int \frac{1}{x-2}\, d(x-2) \quad \left(\text{because } \frac{d(x-2)}{dx} = 1\right) \\ &= \ln|x-2| + C. \\ \int \frac{x}{x^2+1}\, dx &= \int \frac{1}{x^2+1}\, d\left(\frac{1}{2}x^2\right) \quad \left(\text{because } \frac{d(\frac{1}{2}x^2)}{dx} = x\right) \\ &= \frac{1}{2}\int \frac{1}{x^2+1}\, d(x^2+1) \\ &= \frac{1}{2}\ln|x^2+1| + C.\end{aligned}$$

6.3 Integrating $\sin^m x \cos^n x$

To evaluate:

$$\int \sin^m x \cos^n x \, dx,$$

where m, n are non-negative integers, we consider three different cases.

Case 1: m **is odd (**$m = 2p+1$**)** In this case, we use the substitution $u = \cos x$:

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^{2p+1} x \cos^n x \, dx \\
&= -\int \sin^{2p} x \cos^n x \, d(\cos x) \\
&= -\int (1 - \cos^2 x)^p \cos^n x \, d(\cos x) \\
&= -\int (1 - u^2)^p u^n \, du
\end{aligned}$$

Case 2: n **is odd (**$n = 2q + 1$**)** In this case, we use the substitution $u = \sin x$:

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^m x \cos^{2q+1} x \, dx \\
&= \int \sin^m x \cos^{2q} x \, d(\sin x) \\
&= \int \sin^m x (1 - \sin^2 x)^q \, d(\sin x) \\
&= \int u^m (1 - u^2)^q \, du
\end{aligned}$$

Case 3: m, n **are both even (**$m = 2p, n = 2q$**)** In this case, we use half-angle formulas to reduce the powers:

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^{2p} x \cos^{2q} x \, dx \\
&= \int (\sin^2 x)^p (\cos^2 x)^q \, dx \\
&= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^p \left(\frac{1}{2}(1 + \cos 2x)\right)^q \, dx \\
&= \frac{1}{2^{p+q}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q \, dx.
\end{aligned}$$

(Notice that the integrand is a polynomial of $\cos 2x$ with degree $p+q = \frac{1}{2}(m+n)$.)

Example 6.9.

- $$\int \sin^2 x\, dx$$

- $$\int \cos^3 x\, dx$$

- $$\int \sin^5 x\, dx$$

6.4 Integration by Trigonometric Substitution

The idea is as follows:

- Use the substitution $x = a\sin t$ when "$a^2 - x^2$ " occurs (because $1 - \sin^2 t = \cos^2 t$).
- Use the substitution $x = a\tan t$ when "$a^2 + x^2$ " occurs (because $1 + \tan^2 t = \sec^2 t$).
- Use the substitution $x = a\sec t$ when "$x^2 - a^2$ " occurs (because $\sec^2 t - 1 = \tan^2 t$).

Example 6.10.

$$\begin{aligned}
\int \sqrt{2 - x^2}\, dx &= \int \sqrt{2 - (\sqrt{2}\sin t)^2}\, d(\sqrt{2}\sin t) \quad \text{(by letting } x = \sqrt{2}\sin t\text{)} \\
&= \int \sqrt{2}\cos t(\sqrt{2}\cos t\, dt) \\
&= \int 2\cos^2 t\, dt \\
&= \int (\cos 2t + 1)\, dt \\
&= \frac{1}{2}\sin 2t + t + C \\
&= \frac{1}{2}\sin\left(2\arcsin\left(\frac{x}{\sqrt{2}}\right)\right) + \arcsin\left(\frac{x}{\sqrt{2}}\right) + C \\
&= \frac{1}{2}x\sqrt{2 - x^2} + \arcsin\left(\frac{x}{\sqrt{2}}\right) + C.
\end{aligned}$$

(In this course, when handling indefinite integrals, we usually assume t lies in an appropriate region so that $\sqrt{\cos^2 t} = \cos t$, etc., for simplicity.)

Example 6.11.

$$\int \frac{1}{(x^2+x+1)^2}\,dx$$

6.5 Integration by Partial Fractions

Definition 6.12. A rational function $\frac{r}{s}$, where r, s are polynomials, is said to be **proper** if:

$$\deg r < \deg s.$$

By performing long division of polynomials, any rational function $\frac{p}{q}$, where p, q are polynomials, may be expressed in the form:

$$\frac{p}{q} = g + \frac{r}{q},$$

where g is a polynomial, and $\frac{r}{q}$ is a proper rational function.

Let $\frac{r}{s}$ be a proper rational function. Factor s as a product of powers of distinct irreducible factors:

$$s = \cdots (x-a)^m \cdots (\underbrace{x^2+bx+c}_{\text{irreducible i.e. } b^2-4c<0})^n \cdots.$$

Then:

Fact 6.13. *The proper rational function $\frac{r}{s}$ may be written as a sum of rational functions as follows:*

$$\begin{aligned}\frac{r}{s} = \cdots & \\ & + \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} + \cdots \\ & + \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n} \\ & + \cdots,\end{aligned}$$

where the A_i, B_i, C_i are constants.

Example 6.14. $\int \dfrac{x^3 - x - 2}{x^2 - 2x}\, dx$

Performing long division for polynomials, we have:

$$\begin{aligned}\int \frac{(x^3 - x - 2)}{x^2 - 2x} dx &= \int (x+2) dx + \int \frac{3x-2}{x^2-2x} dx \\ &= \frac{1}{2}x^2 + 2x + \int \frac{3x-2}{x^2-2x} dx.\end{aligned}$$

To evaluate:

$$\int \frac{3x-2}{x^2-2x} dx,$$

we first observe that the integrand is a proper rational function. Moreover, the denominator factors as follows:

$$x^2 - 2x = x(x-2).$$

Hence, by Fact 6.13 , we have:

$$\frac{3x-2}{x^2-2x} = \frac{A}{x} + \frac{B}{x-2},$$

for some constants A and B. Clearing denominators, we see that the equation above holds if and only if:

$$3x - 2 = A(x-2) + Bx. \qquad (*)$$

Letting $x = 2$, we have:

$$3 \cdot 2 - 2 = B \cdot 2,$$

which implies that $B = 2$. Similarly, letting $x = 0$ in equation $(*)$ gives:

$$-2 = -2A,$$

which implies that $A = 1$. Hence:

$$\begin{aligned}\int \frac{3x-2}{x^2-2x} dx &= \int \left(\frac{1}{x} + \frac{2}{x-2}\right) dx \\ &= \ln|x| + 2\ln|x-2| + C,\end{aligned}$$

where C represents an arbitrary constant.

We conclude that:

$$\int \frac{(x^3 - x - 2)}{x^2 - 2x} dx \quad = \quad \frac{1}{2}x^2 \ + \ 2x \ + \ \ln|x| \ + \ 2\ln|x-2| \ + \ C.$$

Example 6.15. $\displaystyle\int \frac{x}{(x^2+4)(x-3)}\,dx$

First we note that the integrand is a proper rational function.

The quadratic factor x^2+4 has discriminant $0^2 - 4\cdot 4 < 0$, hence it is irreducible.

By Fact 6.13 , we have:

$$\frac{x}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3},$$

for some constants A, B and C. Clearing denominators, the equation above holds if and only if:

$$x = (Ax+B)(x-3) + C(x^2+4) \tag{$*$}$$

Letting $x = 3$, we have:

$$3 = C\cdot 13,$$

which implies that $C = 3/13$.

Letting $x = 0$, we have:

$$0 = -3B + 4C,$$

which implies that $B = (4/3)C = 4/13$.

Finally, viewing each side of equation $(*)$ as polynomials and comparing the coefficients of x^2 on each side, we have:

$$0 = A + C,$$

which implies that $A = -C = -3/13$.

Hence:

$$\begin{aligned}
&\int \frac{x}{(x^2+4)(x-3)}\,dx \\
&= \frac{1}{13}\int \frac{-3x+4}{x^2+4}\,dx + \frac{3}{13}\int \frac{1}{x-3}\,dx \\
&= \frac{1}{13}\left(\frac{-3}{2}\int \frac{1}{x^2+4}\,d(x^2+4) + \int \frac{1}{(x/2)^2+1}\,dx \right.\\
&\quad \left. +3\int \frac{1}{x-3}\,dx\right) \\
&= \frac{1}{13}\left(\frac{-3}{2}\ln\left|x^2+4\right| + 2\arctan(x/2) + 3\ln|x-3|\right) + D,
\end{aligned}$$

where D represents an arbitrary constant.

Example 6.16. To evaluate

$$\int \frac{x^3+6x+1}{(x^2-1)^2(x^2+1)}\,dx,$$

we apply partial fraction decomposition:

$$\frac{x^3+6x+1}{(x^2-1)^2(x^2+1)} = \frac{x^3+6x+1}{(x-1)^2(x+1)^2(x^2+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1}$$

for some constants A, B, C, D, E, F. By some tedious computations,

$$A = 1,\ B = -\frac{7}{8},\ C = -\frac{3}{4},\ D = -\frac{3}{8},\ E = \frac{5}{4},\ F = \frac{1}{4}.$$

Moreover,

$$\begin{aligned}
\int \frac{1}{x \pm 1}\,dx &= \ln|x \pm 1| + C', \\
\int \frac{1}{(x\pm 1)^2}\,dx &= \int (x \pm 1)^{-2}\,dx = -(x\pm 1)^{-1} + C', \\
\int \frac{1}{x^2+1}\,dx &= \arctan x + C', \\
\int \frac{x}{x^2+1}\,dx &= \int \frac{1}{x^2+1}\,d\left(\frac{1}{2}x^2\right) = \frac{1}{2}\int \frac{1}{x^2+1}\,d\left(x^2+1\right) \\
&= \frac{1}{2}\ln|x^2+1| + C'.
\end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
&\int \frac{x^3+6x+1}{(x^2-1)^2(x^2+1)}\,dx \\
&= -A(x-1)^{-1} + B\ln|x-1| - C(x+1)^{-1} + D\ln|x+1| + \frac{E}{2}\ln|x^2+1| + F\arctan x + C'.
\end{aligned}$$

Example 6.17.

$$\int \frac{x^3}{x^2-1}\,dx$$

By our procedure, we can integrate any rational function as long as we can integrate the building blocks:

$$\frac{1}{x-b},\ \frac{1}{(x-b)^n},\ \frac{x}{x^2+cx+d},\ \frac{1}{x^2+cx+d},\ \frac{x}{(x^2+cx+d)^m},\ \frac{1}{(x^2+cx+d)^m}$$

where $n, m \geq 2$. We have handled the first four in the above example (when $c = 0, d = 1$). For general c, d, we need to complete the square.

Example 6.18.

$$\begin{aligned}\int \frac{x+2}{x^2+2x+2}\,dx &= \int \frac{x+2}{(x+1)^2+1}\,dx \\ &= \int \frac{u+1}{u^2+1}\,du \quad \text{(by letting } u = x+1\text{)} \\ &= \int \frac{u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du \\ &= \frac{1}{2}\ln|u^2+1| + \arctan u + C \\ &= \frac{1}{2}\ln|(x+1)^2+1| + \arctan(x+1) + C.\end{aligned}$$

Finally, when $m \geq 1$, we can still complete the square to force the form $c = 0, d = 1$.

Example 6.19. By trigonometric substitution, evaluate

$$\int \frac{1}{(x^2+1)^2}\,dx.$$

Example 6.20. $\displaystyle\int \frac{x^3}{(x^2+x+1)(x-3)^2}\,dx$

First, we observe that:

$$\frac{x^3}{(x^2+x+1)(x-3)^2}$$

is a proper rational function. Moreover, since the discriminant of x^2+x+1 is $1^2 - 4 < 0$, this quadratic factor is irreducible. So, there exist constants A, B, C, D such that:

$$\frac{x^3}{(x^2+x+1)(x-3)^2} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-3} + \frac{D}{(x-3)^2}.$$

The equation above holds if and only if:

$$\begin{aligned}x^3 = {} & (Ax+B)(x-3)^2 + C(x^2+x+1)(x-3) \\ & + D(x^2+x+1).\end{aligned} \tag{$*$}$$

Letting $x = 3$, we have:

$$27 = 13D.$$

So, $D = 27/13$.

To find A, B and C, we view each side of the equation $(*)$ as polynomials, then compare the coefficients of the x^3, x^2, x and constant terms respectively:

$$x^3: \quad 1 = A + C \tag{6.1}$$
$$x^2: \quad 0 = -6A + B - 2C + 27/13 \tag{6.2}$$
$$x: \quad 0 = 9A - 6B - 2C + 27/13 \tag{6.3}$$
$$1: \quad 0 = 9B - 3C + 27/13 \tag{6.4}$$

Subtracting equation (6.2) from equation (6.3), we have:

$$0 = 15A - 7B,$$

which implies that $B = 15A/7$. Combining this with equation (6.1), we have:

$$B = 15(1 - C)/7 = 15/7 - 15C/7.$$

It now follows from equation (6.4) that:

$$0 = 135/7 - 135C/7 - 3C + 27/13.$$

Hence:

$$C = \frac{162}{169}$$
$$B = \frac{15}{169}$$
$$A = \frac{7}{169}$$
$$D = \frac{27}{13}.$$

We have:

$$\int \frac{x^3}{(x^2 + x + 1)(x - 3)^2}\, dx$$
$$= \int \left[\frac{7x + 15}{169\,(x^2 + x + 1)} + \frac{162}{169\,(x - 3)} + \frac{27}{13\,(x - 3)^2}\right] dx$$

$$= \int \frac{7x + 15}{169\,(x^2 + x + 1)} dx$$
$$+ \frac{162}{169} \int \frac{1}{(x - 3)} dx + \frac{27}{13} \int \frac{1}{(x - 3)^2} dx$$

To evaluate $\int \frac{7x+15}{169(x^2+x+1)} dx$, we first rewrite the integral as follows:

$$\int \frac{7x+15}{169\left(x^2+x+1\right)} dx = \frac{1}{169} \int \frac{7x+7/2-7/2+15}{x^2+x+1} dx$$

$$= \frac{1}{169} \left[\frac{7}{2} \underbrace{\int \frac{2x+1}{x^2+x+1} dx}_{\int \frac{1}{x^2+x+1} d(x^2+x+1)} + \frac{23}{2} \underbrace{\int \frac{1}{(x+1/2)^2+3/4} dx}_{\frac{4}{3} \int \frac{1}{\left((2x+1)/\sqrt{3}\right)^2+1} dx} \right]$$

$$= \frac{7}{338} \ln\left|x^2+x+1\right| + \frac{23 \cdot 2}{169 \cdot 3} \frac{\sqrt{3}}{2} \arctan\left((2x+1)/\sqrt{3}\right) + E$$

$$= \frac{7}{338} \ln\left|x^2+x+1\right| + \frac{23}{169\sqrt{3}} \arctan\left((2x+1)/\sqrt{3}\right) + E,$$

where E represents an arbitrary constant.

It now follows that:

$$\int \frac{x^3}{(x^2+x+1)(x-3)^2} dx$$
$$= \frac{7}{338} \ln\left|x^2+x+1\right| + \frac{23}{169\sqrt{3}} \arctan\left((2x+1)/\sqrt{3}\right)$$
$$+ \frac{162}{169} \ln|x-3| - \frac{27}{13} \frac{1}{x-3} + E.$$

Example 6.21. $\displaystyle\int \frac{8x^2}{x^4+4} dx$

6.6 Integration by Rationalization

Example 6.22. To evaluate

$$\int \frac{\sqrt{x}}{x+1} dx,$$

we let $u = \sqrt{x}$. Since

$$u = \sqrt{x} \implies x = u^2 \implies dx = 2u\, du,$$

$$
\begin{aligned}
\int \frac{\sqrt{x}}{x+1}\,dx &= \int \frac{u}{u^2+1}(2u\,du)\\
&= \int \frac{2u^2}{u^2+1}\,du \quad \text{(which is a rational function now)}\\
&= 2\int \left(1 - \frac{1}{u^2+1}\right) du\\
&= 2u - 2\arctan u + C\\
&= 2\sqrt{x} - 2\arctan\sqrt{x} + C.
\end{aligned}
$$

Example 6.23. To evaluate

$$\int \frac{\sqrt{x}}{\sqrt[3]{x}+1}\,dx,$$

with some thoughts, it's not hard to see that we should use $u = x^{\frac{1}{6}}$. Then, similarly,

$$u = x^{\frac{1}{6}} \implies x = u^6 \implies dx = 6u^5\,du.$$

Hence,

$$
\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt[3]{x}+1}\,dx &= \int \frac{6u^8}{u^2+1}\,du\\
&= 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right) du\\
&= 6\left(\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \arctan u\right) + C\\
&= 6\left(\frac{1}{7}x^{\frac{7}{6}} - \frac{1}{5}x^{\frac{5}{6}} + \frac{1}{3}x^{\frac{3}{6}} - x^{\frac{1}{6}} + \arctan(x^{\frac{1}{6}})\right) + C.
\end{aligned}
$$

6.7 Integrating Basic Trigonometric Functions

Proposition 6.24.

$$
\begin{aligned}
&\int \sin x\,dx = -\cos x + C\\
&\int \cos x\,dx = \sin x + C\\
&\int \tan x\,dx = \ln|\sec x| + C
\end{aligned}
$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$
$$\int \cot x \, dx = -\ln|\csc x| + C$$

Proof of Proposition 6.24.

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= \int \frac{1}{\cos x} \, d(-\cos x) \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

$\cot x$ can be handled similarly.

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\cos x}{\cos^2 x} \, dx \\ &= \int \frac{1}{1-\sin^2 x} \, d(\sin x) \\ &= \int \frac{1}{1-u^2} \, du \quad \text{(by letting } u = \sin x\text{)} \\ &= \frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du \\ &= \frac{1}{2} \ln\left| \frac{1+\sin x}{1-\sin x} \right| + C \\ &= \ln\left| \frac{1+\sin x}{\cos x} \right| + C.\end{aligned}$$

$\csc x$ can be handled similarly. □

6.8 Integration by t-Substitution

Suppose we want to evaluate

$$\int \frac{1}{1+\sin x} \, dx.$$

If we let $t = \tan \frac{x}{2}$, then

$$\begin{aligned}\sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2t}{1+t^2} \\ \cos x &= \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2}(1-t^2) = \frac{1-t^2}{1+t^2}\end{aligned}$$

Furthermore,

$$x = 2 \arctan t \implies dx = \frac{2}{1+t^2}\, dt.$$

Hence,

$$\begin{aligned}\int \frac{1}{1+\sin x}\, dx &= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt \\ &= \int \frac{2}{(t+1)^2}\, dt \\ &= -2(t+1)^{-1} + C \\ &= -2\left(\tan \frac{x}{2} + 1\right)^{-1} + C.\end{aligned}$$

Theorem 6.25 (t-Substitution)**.** *By letting* $t = \tan \frac{x}{2}$, *we have*

$$dx = \frac{2}{1+t^2}\, dt,$$

$$\sin x = \frac{2t}{1+t^2} \quad \textit{and} \ \cos x = \frac{1-t^2}{1+t^2}.$$

With t-substitution, we can transform any rational functions of trigonometric functions into an algebraic rational functions, which can then be handled by partial fractions.

Example 6.26.

$$\int \frac{1}{\sin x + 2\cos x + 1}\, dx.$$

6.9 Integration by Parts

Theorem 6.27 (Integration by Parts)**.**

$$\int u\, dv = uv - \int v\, du.$$

Proof of Integration by Parts. From product rule,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$\int \frac{d}{dx}(uv)\,dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$
$$uv + C = \int v\,du + \int u\,dv$$
$$\int u\,dv = uv - \int v\,du$$

(C can be omitted because of the remained indefinite integrals) □

Example 6.28. To evaluate

$$\int xe^x\,dx,$$

we let $u = x$ and $v = e^x$. Then, $dv = e^x\,dx$ and we have

$$\begin{aligned}\int xe^x\,dx &= \int x\,d(e^x)\\ &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C.\end{aligned}$$

Example 6.29. To evaluate:

$$\int x^2\cos x\,dx,$$

we let $u = x^2$ and $v = \sin x$. Then, $dv = \cos x\,dx$ and we have

$$\begin{aligned}\int x^2\cos x\,dx &= \int x^2\,d(\sin x)\\ &= x^2\sin x - \int \sin x\,d(x^2)\\ &= x^2\sin x - \int 2x\sin x\,dx.\end{aligned}$$

So, to apply integration by parts once, we practically integrate $\cos x$ once and differentiate x^2 once. That's how to determine which function to be u and which function to be v. We then proceed to apply integration by parts one more time:

$$\begin{aligned}\int x^2 \cos x\, dx &= x^2 \sin x - \int 2x \sin x\, dx \\ &= x^2 \sin x + 2\int x\, d(\cos x) \\ &= x^2 \sin x + 2x\cos x - 2\int \cos x\, dx \\ &= x^2 \sin x + 2x \cos x - 2\sin x + C.\end{aligned}$$

Example 6.30. How about

$$\int \arcsin x\, dx?$$

We don't know how to integrate $\arcsin x$, but we know how to differentiate it. So,

$$\begin{aligned}\int \arcsin x\, dx &= x \arcsin x - \int x d(\arcsin x) \\ &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx \\ &= x \arcsin x - \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\, d(\sin\theta) \quad \text{(by letting } x = \sin\theta\text{)} \\ &= x\arcsin x - \int \sin\theta\, d\theta \\ &= x \arcsin x + \cos\theta + C \\ &= x\arcsin x + \sqrt{1-x^2} + C.\end{aligned}$$

Example 6.31.

- $$\int \ln x\, dx$$

- $$\int e^x \sin x\, dx$$

6.10 Reduction Formula

Example 6.32. How to evaluate:

$$\int x^4 e^x\, dx\,?$$

Instead of applying integration by parts four times, we could set up a **reduction formula** as follows.

Let

$$I_n = \int x^n e^x \, dx,$$

where n is a non-negative integer. By integration by parts,

$$\begin{aligned}\int x^n e^x \, dx &= \int x^n \, d(e^x) \\ &= x^n e^x - \int e^x \, d(x^n) \\ &= x^n e^x - n \int x^{n-1} e^x \, dx\end{aligned}$$

provided that $n \geq 1$. In other words,

$$I_n = x^n e^x - n I_{n-1} \quad \text{for all } n \geq 1.$$

All we need now would be the initial result:

$$I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + C.$$

We can then easily generate I_n up to any n using our reduction formula:

$$\begin{aligned}&I_1 = x^1 e^x - I_0 = xe^x - e^x + C \\ &I_2 = x^2 e^x - 2I_1 = x^2 e^x - 2xe^x + 2e^x + C \\ &I_3 = x^3 e^x - 3I_2 = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C \\ &I_4 = x^4 e^x - 4I_3 = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C.\end{aligned}$$

Example 6.33. Let:

$$I_n = \int \frac{1}{x^n (x+1)} \, dx,$$

where n is a non-negative integer. This problem can be solved by partial fraction decomposition if n is given. Interestingly, we can also set up a reduction formula as follows. For $n \geq 1$,

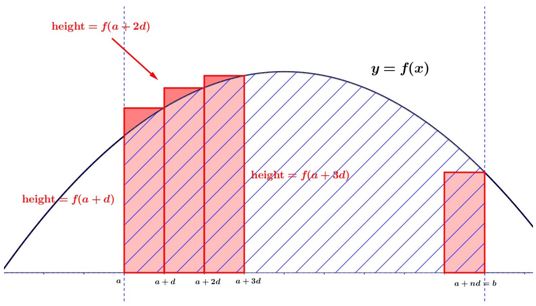
$$\begin{aligned} I_n &= \int \frac{1}{x^n(x+1)}\, dx \\ &= \int \left(\frac{1+x}{x^n(x+1)} - \frac{x}{x^n(x+1)} \right) dx \\ &= \int \frac{1}{x^n}\, dx - \int \frac{1}{x^{n-1}(x+1)}\, dx \\ &= \begin{cases} \ln|x| - I_0 & \text{if } n = 1; \\ \dfrac{1}{-n+1} x^{-n+1} - I_{n-1} & \text{if } n \geq 2. \end{cases} \end{aligned}$$

For $n = 0$, we have:

$$I_0 = \int \frac{1}{x+1}\, dx = \ln|x+1| + C.$$

6.11 Definite Integral and Riemann Sum

For a continuous function $f(x)$, how to find the area of the region under the curve $y = f(x)$ over the interval $[a, b]$?



If we cut the interval $[a, b]$ into n parts, then the width of each would be d. The right end-points of the sub-intervals will then be:

$$a + d, a + 2d, a + 3d, \ldots, a + nd = b.$$

So, the heights of the rectangles in the above graph are

$$f(a + d), f(a + 2d), f(a + 3d), \ldots, f(a + nd).$$

Therefore, $\sum_{k=1}^{n} f(a + kd)d$ is the total area of the rectangles. When n goes to infinity, this value will be exactly the (signed) area under the curve $y = f(x)$ over the interval $[a, b]$.

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Definition 6.34. The **definite integral** of a piecewise continuous function $f(x)$ over an interval $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(a+kd)d \quad \text{where } d = \frac{b-a}{n}$$

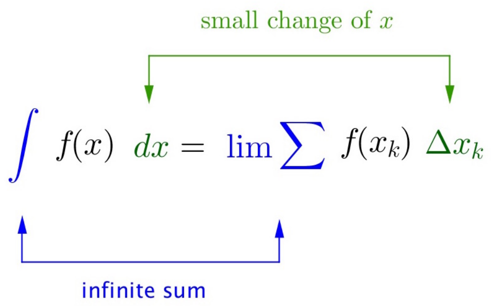
Proof of Definition 6.34. For its well-definedness, see Definition 1 and Theorem 1 in Appendix 5. □

It will be convenient to extend our definition to arbitrary a, b :

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{if } a > b.$$

Notice that if we let $x_k = a + kd$ be the right end-points of the sub-intervals, then

$$\Delta x_k = x_k - x_{k-1} = d$$



That explains why we keep "dx " as part of the notation.

Example 6.35. Consider the function $f(x) = x^2$ over the interval $[a, b]$. By definition,

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{n\to+\infty} \sum_{k=1}^{n} f(a+kd)d \quad \text{where } d = \frac{b-a}{n} \\
&= (b-a) \lim_{n\to+\infty} \frac{1}{n}((a+d)^2 + (a+2d)^2 + \cdots + (a+nd)^2) \\
&= (b-a) \lim_{n\to+\infty} \frac{1}{n}((a^2+2ad+d^2) + (a^2+4ad+4d^2) + \cdots + (a^2+2and+n^2d^2)) \\
&= (b-a) \lim_{n\to+\infty} \frac{1}{n}\left(na^2 + 2ad\,\frac{n(n+1)}{2} + d^2\,\frac{n(n+1)(2n+1)}{6}\right) \\
&= (b-a) \lim_{n\to+\infty} \frac{1}{n}\left(na^2 + 2a\,\frac{b-a}{n}\,\frac{n(n+1)}{2} + \frac{(b-a)^2}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\
&= (b-a)\left(a^2 + 2a(b-a)\frac{1}{2} + (b-a)^2\frac{2}{6}\right) \\
&= \frac{1}{3}(b^3-a^3).
\end{aligned}$$

Proposition 6.36. *For any constants* $a, b, c, \alpha, \beta \in \mathbb{R}$*,*

$$\begin{aligned}
\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx; \\
\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx; \\
\int_a^b (\alpha f(x) + \beta g(x))\,dx &= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx; \\
f(x) \le g(x) \text{ on } [a,b] \implies & \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.
\end{aligned}$$

Proof of Proposition 6.36. See Proposition 2 in Appendix 5. □

6.12 Fundamental Theorem of Calculus

For a continuous function $f(t)$, we may define a function:

$$F(x) := \int_a^x f(t)\,dt$$

, where a is any element in the domain of $f(x)$.

We are now ready to state and prove the following fundamental results in Calculus, which basically mean

- "Integration and differentiation are reverse of each other."
- "Definite integrals can be computed by indefinite integrals."

Theorem 6.37 (Fundamental theorem of calculus FTC)**.**

- **Part I:** *If $f(x)$ is a continuous function on $[a, b]$, then the function $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$ and:*

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

for all $x \in [a, b]$.

- **Part II:** *If $F(x)$ is a differentiable function on $[a, b]$ and $F'(x)$ is continuous on $[a, b]$, then:*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Proof of Fundamental theorem of calculus (FTC). First: For any $x \in (a, b)$ and small $h > 0$, by EVT, we can define $m(h)$ and $M(h)$ such that f attains its minimum and maximum within $[x, x + h]$ respectively. Then,

$$RF'(x) = \lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0^+} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

Since

$$f(m(h)) = \frac{\int_x^{x+h} f(m(h))\,dt}{h} \leq \frac{\int_x^{x+h} f(t)\,dt}{h} \leq \frac{\int_x^{x+h} f(M(h))\,dt}{h} = f(M(h))$$

and, by continuity,

$$\lim_{h\to 0^+} f(m(h)) = f(x) = \lim_{h\to 0^+} f(M(h)),$$

we can conclude that $RF'(x) = f(x)$ (squeeze theorem). Similarly,

$$LF'(x) = \lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0^-} \frac{F(x) - F(x - |h|)}{|h|}$$

By applying the same arguments over $[x - |h|, x]$, we can also conclude that $LF'(x) = f(x)$. Therefore, F is differentiable and

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = F'(x) = f(x) \quad \text{for all } x \in (a, b).$$

At $x = a$, by continuity and squeeze theorem, we also have

$$RF'(a) = \lim_{h \to 0^+} \frac{F(a+h) - F(a)}{h} = \lim_{h \to 0^+} \frac{\int_a^{a+h} f(t)\, dt}{h} = f(a)$$

Obviously, $LF'(b) = f(b)$ as well. Second: Suppose $g(x)$ is differentiable and $g'(x) = 0$ on $[a, b]$. By Lagrange's MVT, for any $x \in (a, b]$,

$$\frac{g(x) - g(a)}{x - a} = g'(c) = 0 \quad \text{for some } c \implies g(x) = g(a)$$

Therefore, $g(x)$ must be a constant function on $[a, b]$.

Now, by the first part of FTC,

$$\frac{d}{dx}\left(F(x) - \int_a^x F'(t)\, dt\right) = F'(x) - \frac{d}{dx}\int_a^x F'(t)\, dt = 0$$

for all $x \in [a, b]$. So,

$$g(x) = F(x) - \int_a^x F'(t)\, dt$$

must be a constant function on $[a, b]$. Hence,

$$F(b) - \int_a^b F'(t)\, dt = g(b) = g(a) = F(a)$$

and the result follows. □

By the second part of FTC, for any continuous function $f(x)$ and $a, b \in \mathbb{R}$, we have:

$$\int_a^b f(x)\, dx = F(x)\big|_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Remark. Replacing $F(x)$ with $F(x)+C$ for any constant C would have no effect, as:

$$(F(b) + C) - (F(a) + C) = F(b) - F(a).$$

Example 6.38. Let us redo Example 6.35 by FTC:

$$\int_a^b f(x)\, dx = \int_a^b x^2\, dx = \frac{1}{3}x^3\bigg|_a^b = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

Example 6.39. Consider the function $y = f(x) = x^3$ over the interval $[-1, 1]$.

Since $f(x) \geq 0$ when $x \in [0, 1]$ and $f(x) \leq 0$ when $x \in [-1, 0]$, to find the area of the region bounded by $y = f(x)$ and the x -axis, we need to split the interval $[-1, 1]$ into $[-1, 0]$ and $[0, 1]$ (to avoid cancellation):

$$\text{Area} = \left| \int_{-1}^{0} x^3 \, dx \right| + \left| \int_{0}^{1} x^3 \, dx \right| = \frac{1}{2}.$$

(In fact, the signed areas of the two regions will cancel out each other:

$$\int_{-1}^{1} x^3 \, dx = 0.)$$

Example 6.40. When applying integration by substitution on definite integrals, there's no need to substitute x back in at the end as long as we adjust the bounds accordingly:

$$\begin{aligned} \int_{0}^{1} (2x+1)^{\frac{3}{7}} \, dx &= \int_{1}^{3} u^{\frac{3}{7}} \left(\frac{1}{2} du \right) \quad \text{(by letting } u = 2x+1) \\ &= \frac{1}{2} \cdot \frac{7}{10} \, u^{\frac{10}{7}} \Big|_{1}^{3} = \frac{1}{2} \cdot \frac{7}{10} (3^{\frac{10}{7}} - 1). \end{aligned}$$

Example 6.41.

- $$\int_{-2}^{-1} \frac{1}{x} \, dx;$$

- $$\int_{-3}^{0} |x^2 + 3x + 2| \, dx;$$

- $$\int_{0}^{\pi} x^6 \sin x \, dx.$$

Here's a surprising application of FTC.

Proposition 6.42. *If $f(x)$ is a continuous function over $[0, 1]$, then:*

$$\int_{0}^{1} f(x) \, dx = \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n} \right)$$

Example 6.43.

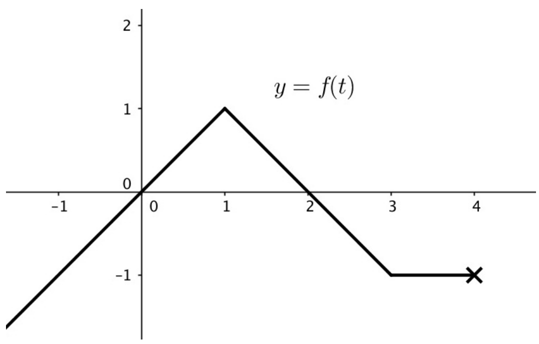
$$\begin{aligned}
&\lim_{n\to+\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+n^2}\right)\\
&=\lim_{n\to+\infty}\frac{1}{n}\left(\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\cdots+\frac{1}{1+\left(\frac{n}{n}\right)^2}\right)\\
&=\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^2}\\
&=\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)
\end{aligned}$$

where $f(x)=\dfrac{1}{1+x^2}$.

Hence,

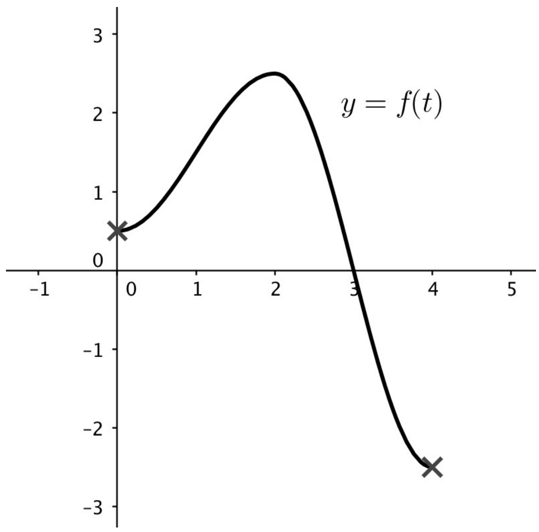
$$\lim_{n\to+\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+n^2}\right)=\int_0^1\frac{1}{1+x^2}\,dx=\frac{\pi}{4}.$$

Example 6.44. Given that $F(x)=\displaystyle\int_2^x f(t)\,dt$ where



Find $F(2), F(4)$ and $F(0)$.

Example 6.45. Given that $F(x)=\displaystyle\int_0^x f(t)\,dt$ where



Among $x \in [0, 4]$,

- When is $F(x)$ maximum?
- When is $F(x)$ minimum?
- When is $F'(x)$ maximum?

We may also generalize the first part of FTC a little bit:

Proposition 6.46. *If $f(x)$ is a continuous on $[a, b]$ and $h(x)$ is differentiable on $[c, d]$ such that $h([c, d]) \subseteq [a, b]$, then*

$$\frac{d}{dx} \int_a^{h(x)} f(t)\, dt = f(h(x))\, h'(x) \quad \textit{for all } x \in [c, d]$$

Proof of Proposition 6.46. Let $F(x) = \displaystyle\int_a^x f(t)\, dt$. Then, by FTC, $F(x)$ is differentiable over $[a, b]$. Moreover, for any $x \in [c, d]$,

$$\frac{d}{dx} \int_a^{h(x)} f(t)\, dt = \frac{d}{dx} F(h(x)) = F'(h(x))h'(x) = f(h(x))h'(x)$$

as desired. □

Example 6.47. Find $g'(x)$ if

- $$g(x) = \int_{-1}^{x^2} \cos(t^2)\, dt;$$

- $$g(x) = \int_{x^4}^{1} \sec(\sqrt[3]{t})\, dt;$$

- $$g(x) = \int_{\sin x}^{\cos x} \ln(\sin t)\, dt.$$

6.13 Improper Integrals

Sometimes, we are interested in computing the definite integral of a function over $[a, +\infty)$ or $(-\infty, b]$.

Definition 6.48. If $f(x)$ is a continuous function over $[a, +\infty)$ such that the limit

$$\lim_{b \to +\infty} \int_a^b f(x)\, dx$$

exists, we define

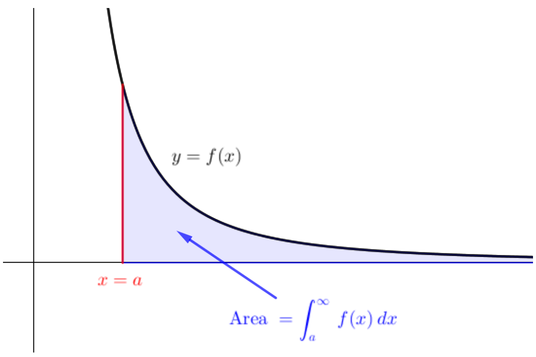
$$\int_a^{+\infty} f(x)\, dx = \lim_{b \to +\infty} \int_a^b f(x)\, dx$$

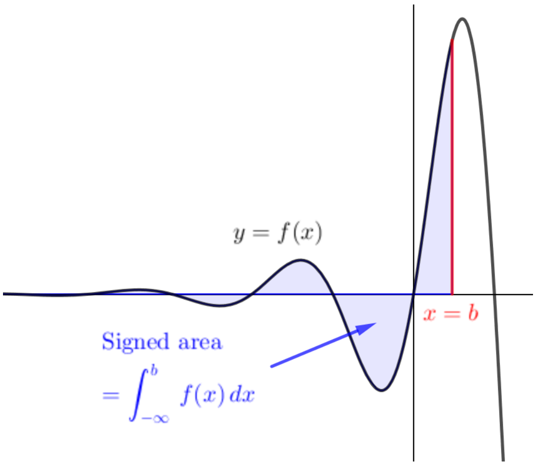
Similarly, if $f(x)$ is a continuous function over $(-\infty, b]$ such that the limit

$$\lim_{a \to -\infty} \int_a^b f(x)\, dx$$

exists, we define

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx$$





Occasionally, we may also consider one-sided **improper integrals**.

Definition 6.49. If $f(x)$ is a continuous function over $(a, b]$ such that the limit

$$\lim_{c \to a^+} \int_c^b f(x)\,dx$$

exists, we define

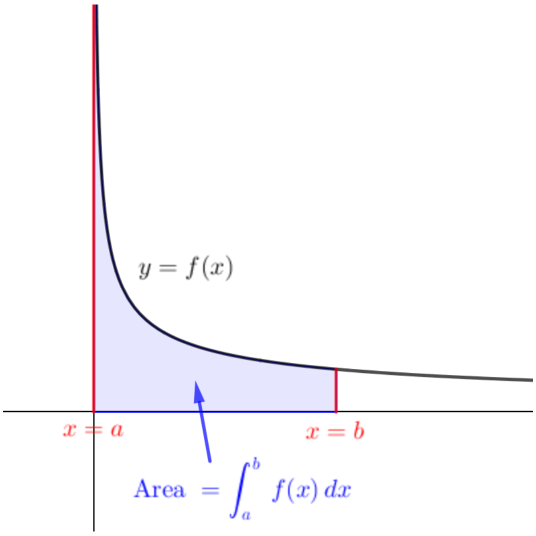
$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

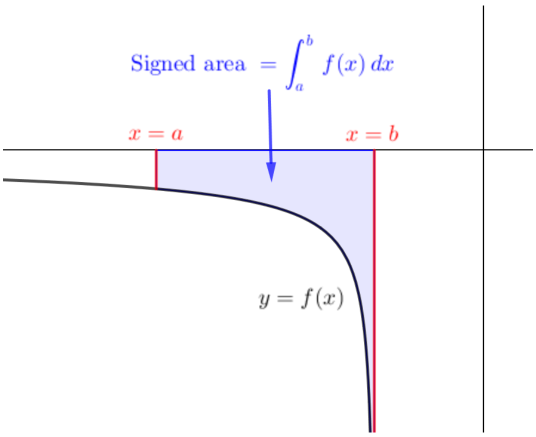
Similarly, if $f(x)$ is a continuous function over $[a, b)$ such that the limit

$$\lim_{c \to b^-} \int_a^c f(x)\,dx$$

exists, we define

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$





In all four cases above, we say that the improper integral **converges** if the corresponding limit exists. Otherwise, we say that it **diverges** .

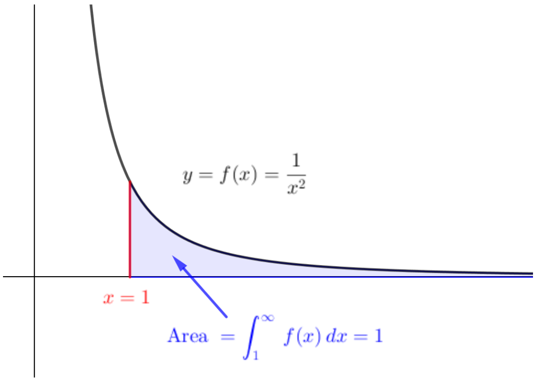
Example 6.50. Since

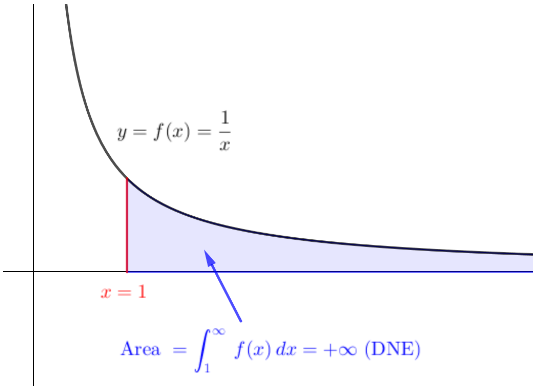
$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to+\infty} -\frac{1}{x}\bigg|_1^b = 1,$$

we can conclude that $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges to 1. On the other hand, since:

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to+\infty} \ln|x|\,\big|_1^b = +\infty \quad \text{(DNE)},$$

we can conclude that $\int_1^\infty \frac{1}{x}\,dx$ diverges to $+\infty$.





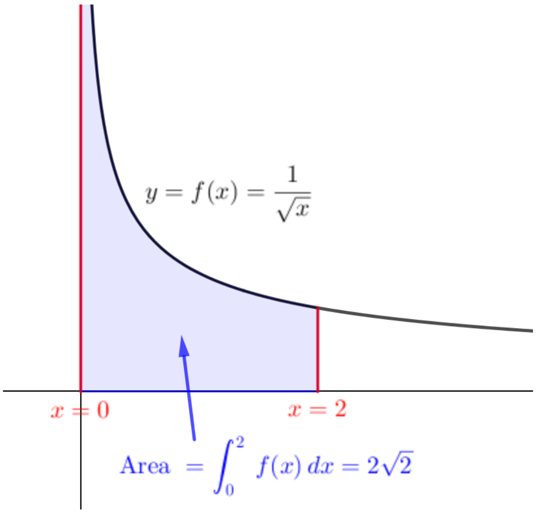
Example 6.51. Since:

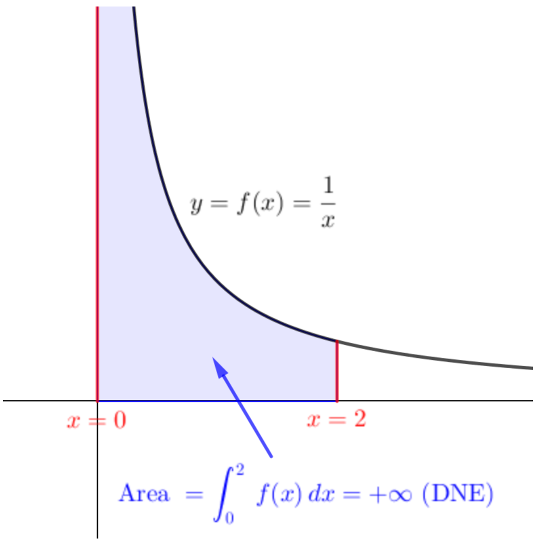
$$\int_0^2 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^2 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^2 = 2\sqrt{2},$$

we can conclude that $\int_0^2 \frac{1}{\sqrt{x}}\,dx$ converges to $2\sqrt{2}$.On the other hand, since

$$\int_0^2 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^2 \frac{1}{x}\,dx = \lim_{c\to 0^+} \ln|x|\,\Big|_c^2 = +\infty \quad \text{(DNE)},$$

we can conclude that $\int_0^2 \frac{1}{x}\,dx$ diverges to $+\infty$.


$y = f(x) = \frac{1}{\sqrt{x}}$
$x = 0$
$x = 2$
Area $= \int_0^2 f(x)\,dx = 2\sqrt{2}$


$y = f(x) = \frac{1}{x}$
$x = 0$
$x = 2$
Area $= \int_0^2 f(x)\,dx = +\infty$ (DNE)