Lecture 7

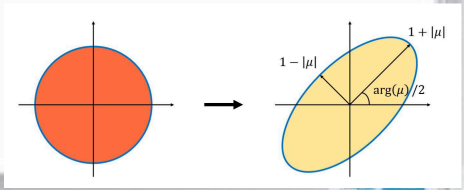
Definition: (Quasiconformal map) Let $f: \mathbb{C} \to \mathbb{C}$ be a C^1 homeomorphism. f is called a quasi-conformal map with respect to a complex-valued function $\mu: \mathbb{C} \to \mathbb{C}$, called the Beltrami coefficient, with $\|\mu\|_\infty < 1$ if:

$$(*) \quad \frac{\partial f}{\partial \bar{z}}(z) = \mu(z) \frac{\partial f}{\partial z} \quad \text{where}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$$

$\mu(z)$ measures the local geometric distortion at z.

$(*)$ is called the Beltrami's equation



Remark: 1. When $\mu \equiv 0$, the Beltrami's equation is reduced to the Cauchy-Riemann equation. Let $f = u + iv$ (u, v real functions).

Then:
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) - i\frac{\partial}{\partial y}(u+iv)\right)$$
$$= \frac{1}{2}\left((u_x + v_y) + i(v_x - u_y)\right) = 0$$

$$\Rightarrow \begin{cases} u_x = -v_y \\ u_y = +v_x \end{cases} \quad \text{(Cauchy-Riemann eqt)}$$

2. In matrix form, a conformal/holomorphic complex-value function $f = u + iv$ satisfies:

$$Df(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

Or $$\begin{pmatrix}-v_y\\ v_x\end{pmatrix} = \overset{\text{Id}}{\begin{pmatrix}+1 & 0\\ 0 & +1\end{pmatrix}}\begin{pmatrix}u_x\\ u_y\end{pmatrix}. \quad \text{———} \quad (**)$$

Quasi-conformal map generalizes (**) by considering

$$\begin{pmatrix}-v_y\\ v_x\end{pmatrix} = \underbrace{\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}}_{\text{Represent the metric distortion}}\begin{pmatrix}u_x\\ u_y\end{pmatrix}$$ for some α, β and γ depending on μ.

3. Let $J(z)$ = Jacobian of $f = u + iv$ at z.

Then $J = \det\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix} = u_x v_y - u_y v_x$

Note that:

$$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 = \frac{(u_x + v_y)^2 + (v_x - u_y)^2}{4} - \frac{(u_x - v_y)^2 + (v_x + u_y)^2}{4}$$

$$= (u_x v_y - u_y v_x) = J(z)$$

$$\therefore J(z) = \left|\frac{\partial f}{\partial z}\right|^2\left(1 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \Big/ \left|\frac{\partial f}{\partial z}\right|^2\right) = \left|\frac{\partial f}{\partial z}\right|^2\left(1 - |\mu(z)|^2\right)$$

Thus, if $\|\mu(z)\|_\infty < 1$ and $\left|\frac{\partial f}{\partial z}\right| \neq 0$ (f = homeomorphism) then $J(z) > 0$ everywhere. $\therefore$ f is orientation-preserving everywhere

Existence and Uniqueness Theorem

Theorem: (Measurable Riemann mapping theorem) Suppose $\mu : \mathbb{C} \to \mathbb{C}$ is Lebesgue measurable and satisfies $\|\mu\|_\infty < 1$, then there exists a quasi-conformal homeomorphism ϕ from $\mathbb{C}$ onto itself, which is in the Sobolev space $W^{1,2}(\mathbb{C})$ and satisfies the Beltrami equation $\left(\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}\right)$ in the distribution sense. Also, by fixing $0, 1, \infty$, the associated quasiconformal homeomorphism ϕ is uniquely determined.

Theorem: Suppose $\mu : \mathbb{D} \to \mathbb{C}$ is Lebesgue measurable and satisfies $\|\mu\|_\infty < 1$. Then, there exists a quasiconformal homeomorphism ϕ from $\mathbb{D}$ to itself, which is in the Sobolev space $W^{1,2}(\Omega)$ and satisfies the Beltrami equation in the distribution sense. Also, by fixing 0 and 1, ϕ is uniquely determined.

Proof: Follows from previous thm by reflection.
(Based on Beltrami holomorphic flow Later!)

Composition of quasiconformal maps

Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be quasiconformal maps. Then, the Beltrami coefficient of the composition map $g \circ f$ is given by:

$$\mu_{g \circ f}(z) = \frac{\mu_f(z) + \overline{f_z(z)}/f_z(z)\,(\mu_g \circ f)}{1 + \overline{f_z(z)}/f_z(z)\,\overline{\mu_f}\,(\mu_g \circ f)} .$$

Theorem: Let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ be quasiconformal maps. Suppose the Beltrami coefficients of f^{-1} and g are the same. Then the Beltrami coefficient of $g \circ f$ is equal to 0 and $g \circ f$ is conformal.

Proof: Note that: $\mu_{f^{-1}} \circ f = -\left(f_z / |f_z| \right) \mu_f$.

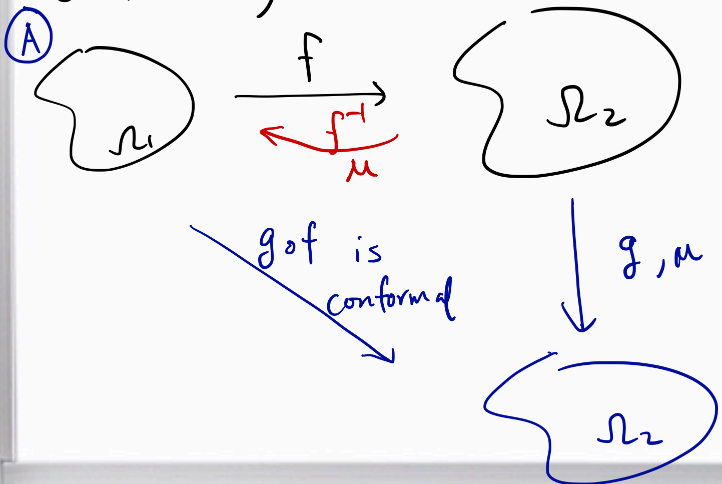
$\because \mu_{f^{-1}} = \mu_g$, we have:

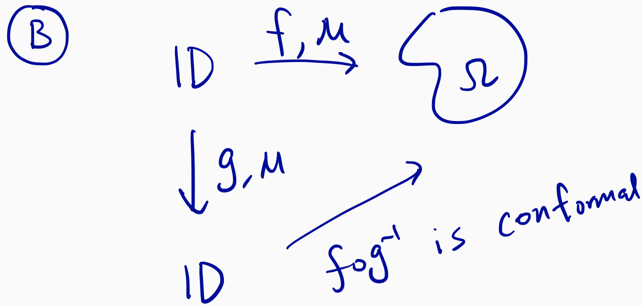
$$\mu_f + \left(\overline{f_z}/f_z\right)(\mu_g \circ f) = \mu_f + \left(\overline{f_z}/f_z\right)(\mu_{f^{-1}} \circ f)$$

$$= \mu_f + \left(\overline{f_z}/f_z\right)\left(-\frac{f_z}{\overline{f_z}}\right)\mu_f = 0$$

By the composition formula, $\mu_{g \circ f} = 0$ and so $g \circ f$ is conformal.

<u>Remark</u>: The above theorem gives a useful way to fix conformality distortion.





In depth analysis of Beltrami's equation

Let $f = u + iv$ and $\mu = \rho + i\tau$. Comparing the real and imaginary parts of $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$ gives:

$$\begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \rho+1 & \tau \\ \tau & 1-\rho \end{pmatrix} \begin{pmatrix} -v_y \\ v_x \end{pmatrix}.$$

$\because \ \|\mu\|_\infty < 1$, $\det \begin{pmatrix} \rho+1 & \tau \\ \tau & 1-\rho \end{pmatrix} = 1 - \rho^2 - \tau^2 > 0$ for $\forall z \in \Omega$.

$$\therefore \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \frac{1}{1-\rho^2-\tau^2} \begin{pmatrix} 1-\rho & -\tau \\ -\tau & \rho+1 \end{pmatrix} \begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Denote $C = \begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix}$. We get $\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \frac{-1}{1-\rho^2-\tau^2} C^T C \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

where

$$-A = \frac{-1}{1-\rho^2-\tau^2} \begin{pmatrix} 1-\rho & -\tau \\ -\tau & \rho+1 \end{pmatrix} \begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix} = \frac{-1}{1-\rho^2-\tau^2} \begin{pmatrix} -(1-\rho)^2-\tau^2 & 2\tau \\ 2\tau & -\tau^2-(\rho+1)^2 \end{pmatrix}$$

Area distortion under quasi-conformal map

To simplify our discussion, let $f: \underbrace{[0,1]\times[0,1]}_{R} \to \Omega \subseteq \mathbb{C}$.

$(\therefore$ Area of source domain R is $1)$

Now, area of $\Omega = \int_R J(z)\, dz$

$$= \int_R (u_x v_y - v_x u_y)\, dz$$

Recall that $\begin{pmatrix} +v_y \\ -v_x \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \alpha u_x + \beta u_y \\ \beta u_x + \gamma u_y \end{pmatrix}$

$\therefore$ Area of $\Omega = \int_R u_x (\alpha u_x + \beta u_y) + (\beta u_x + \gamma u_y) u_y$

$$= \int_R \alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2$$

where α, β and γ are determined by $\mu = \rho + i\tau$.

Remark: • μ (or α, β, γ) introduces area distortion under f

• Computationally, once u associated to μ is obtained, we can determine the area of the target domains by

$$A = \int_R \alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2$$

If $\Omega = [0,1] \times [0,h]$, then $h = A$.

$\therefore$ Once u is computed, the geometry of the target domain can be determined.

$\therefore$ v can be computed (Useful observation!)

Computation of QC maps

Goal: Given μ, our goal is to compute q.c. map f associated to μ.

Method 1: (Simple least square method)

Minimize the residual of the Beltrami's eqt

$$E(f) = \int_\Omega \left| \frac{\partial f}{\partial \bar{z}} - \mu \frac{\partial f}{\partial z} \right|^2 dz$$

Method 2: $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$ $\xrightarrow[\text{formulation}]{\text{variational}}$ Energy minimization model (not least sq. model)

$\Downarrow$

Elliptic PDE

Solving elliptic PDE will give "smoothness" of the solution

Method 3: Beltrami Holomorphic Flow

Goal: given μ, want to get f^{μ} associated to μ.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, leftrightarrow] & \frac{\mu}{N} \arrow[r] \arrow[d, leftrightarrow] & \frac{2\mu}{N} \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \mu \arrow[d, leftrightarrow] \\
\mathrm{Id} \arrow[r] & f^{\frac{\mu}{N}} \arrow[r] & f^{2\mu/N} \arrow[r] & \cdots \arrow[r] & f^{\mu}
\end{tikzcd}$$

(N is big)

If $\tilde{\mu} = \mu + w$ ← small

$$f^{\tilde{\mu}} = f^{\mu} + \vec{V}(w)$$

Simple way to compute QC map

Goal: Given μ, our goal is to compute the associated QC map.

Idea: Minimize the residual of Beltrami's eqt:

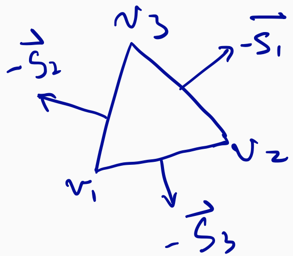
$$E(f) = \iint_{\Omega} \left| \frac{\partial f}{\partial \bar{z}} - \mu \frac{\partial f}{\partial z} \right|^2 dz .$$

Let $M = (V, E, F)$ be triangular mesh and f be piecewise linear function. Choose a face $[v_1, v_2, v_3] \in F$ and embed it on $\mathbb{R}^2$. Let the planar coordinate of v_k be (x_k, y_k).

Let $f : (x,y) \mapsto (u(x,y), v(x,y))$.

As before,

$$\begin{cases} \nabla u = u_1 \vec{S_1} + u_2 \vec{S_2} + u_3 \vec{S_3} \\ \nabla v = v_1 \vec{S_1} + v_2 \vec{S_2} + v_3 \vec{S_3} \end{cases}$$



$u_1 = u(v_1)$, $u_2 = u(v_2)$, etc ...

Then, u_x, u_y, v_x, v_y are constants on each face.

$\therefore$ μ is piecewise constant function on each face T, namely,

$$\mu(T) = \rho_T + i\,\tau_T$$

We can check that the Beltrami's eqt is equivalent to:

$$\begin{cases} \vec{p}_T \cdot \nabla u + \vec{q}_T \cdot \nabla v = 0 \\ -\vec{q}_T \cdot \nabla u + \vec{p}_T \cdot \nabla v = 0 \end{cases} \quad \text{where } \vec{p}_T = \begin{pmatrix} \rho_T - 1 \\ \tau_T \end{pmatrix}, \ \vec{q}_T = \begin{pmatrix} -\tau_T \\ \rho_T + 1 \end{pmatrix}$$

Rewriting:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \quad \text{where}$$

$$\lambda_k = \vec{p} \cdot \vec{S}_k, \quad \gamma_k = \vec{q} \cdot \vec{S}_k, \quad k = 1, 2, 3$$

For each face, we can construct a linear system.

Pack all linear equations together to form a big linear system:

$$\begin{pmatrix} \Lambda & P \\ -P & \Lambda \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \vec{0} \quad \text{where } \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} ; \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

coordinate values at different vertices.

The big linear system is solved by minimizing: $\left\| \begin{pmatrix} \Lambda & P \\ -P & \Lambda \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \right\|^2$

subject to some boundary conditions! (Standard least square problem)

<u>Remark</u>: The method is equivalent to minimizing:

$$\int_\Omega \left| \frac{\partial f}{\partial \bar{z}} - \mu \frac{\partial f}{\partial z} \right|^2 \quad \text{(Least square Beltrami energy)}$$

<u>Drawback</u>: 1. May get trapped in local minimum

2. Resulting map may just be immersion (w/ self-overlap)

Another method to solve Beltrami's equation

Linear Beltrami Solver (LBS)

Let $M = (V, E, F)$ be simply-connected domain w/ boundary.

Let $V = \{(g_1, h_1), (g_2, h_2), \ldots, (g_{|V|}, h_{|V|})\}$.

In discrete formulation, given $\mu = \rho + i\tau$, we want to compute a resulting mesh M' such that

$$v_n = (g_n, h_n) \mapsto w_n = (s_n, t_n)$$ vertices in M'

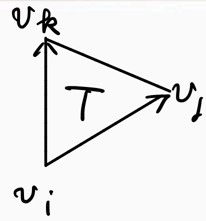
On each face T, the discrete QC map f is linear.

$$\therefore \quad f|_T(x,y) = \begin{pmatrix} u|_T(x,y) \\ v|_T(x,y) \end{pmatrix} = \begin{pmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + s_T \end{pmatrix}$$

"$u + iv$"

$$\therefore \quad u_x|_T = a_T \; ; \quad u_y|_T = b_T \; ; \quad v_x|_T = c_T \; ; \quad v_y|_T = d_T$$

Consider the directional derivatives along $v_j - v_i$ and $v_k - v_i$, we get:



$$\begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \begin{pmatrix} g_j - g_i & g_k - g_i \\ h_j - h_i & h_k - h_i \end{pmatrix} = \begin{pmatrix} s_j - s_i & s_k - s_i \\ t_j - t_i & t_k - t_i \end{pmatrix}$$

Assume f is orientation-preserving, then:

$$\det \begin{pmatrix} g_j - g_i & g_k - g_i \\ h_j - h_i & h_k - h_i \end{pmatrix} = 2\,\mathrm{Area}(T).$$

$$\therefore \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} = \frac{1}{2\,\mathrm{Area}(T)} \begin{pmatrix} s_j - s_i & s_k - s_i \\ t_k - t_i & t_k - t_i \end{pmatrix} \begin{pmatrix} h_k - h_i & g_i - g_k \\ h_i - h_j & g_j - g_i \end{pmatrix}$$

$$\begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} = \begin{pmatrix} A_T^i s_i + A_T^j s_j + A_T^k s_k & B_T^i s_i + B_T^j s_j + B_T^k s_k \\ A_T^i t_i + A_T^j t_j + A_T^k t_k & B_T^i t_i + B_T^j t_j + B_T^k t_k \end{pmatrix}$$

$$\begin{bmatrix} a_T & b_T \\ c_T & d_T \end{bmatrix} = \frac{1}{2 \cdot Area(T)} \begin{bmatrix} s_j - s_i & s_k - s_i \\ t_j - t_i & t_k - t_i \end{bmatrix} \begin{bmatrix} h_k - h_i & g_i - g_k \\ h_i - h_j & g_j - g_i \end{bmatrix}$$
$$= \begin{bmatrix} A_T^i s_i + A_T^j s_j + A_T^k s_k & B_T^i s_i + B_T^j s_j + B_T^k s_k \\ A_T^i t_i + A_T^j t_j + A_T^k t_k & B_T^i t_i + B_T^j t_j + B_T^k t_k \end{bmatrix}.$$

where

$$A_T^i = (h_j - h_k)/2 \cdot Area(T); \quad A_T^j = (h_k - h_i)/2 \cdot Area(T); \quad A_T^k = (h_i - h_j)/2 \cdot Area(T);$$
$$B_T^i = (g_k - g_j)/2 \cdot Area(T); \quad B_T^j = (g_i - g_k)/2 \cdot Area(T); \quad B_T^k = (g_j - g_i)/2 \cdot Area(T).$$

Next time, we will define: discrete divergence such that

- $\mathrm{Div}(\underbrace{-d_T}_{= -v_y|_T}, \underbrace{c_T}_{= v_x|_T}) = 0$
- $\mathrm{Div}(\underbrace{-b_T}_{= -u_y|_T}, \underbrace{a_T}_{= u_x|_T}) = 0$

With that

$$0 = \mathrm{Div}\left(\begin{pmatrix} -d_T \\ c_T \end{pmatrix}\right) = \mathrm{Div}\left(\begin{pmatrix} \alpha_T & \beta_T \\ \beta_T & \gamma_T \end{pmatrix}\begin{pmatrix} a_T \\ b_T \end{pmatrix}\right)$$