MMAT 5390: Mathematical Image Processing

Assignment 5

Due: April 24, 2024

Please give reasons in your solutions.

1. f is a $N \times N$ image and $\vec{f}$ is its vectorized image. The constrained least square filtering aims to

$$\min_{\vec{f}} E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$
$$\text{s.t. } [\vec{g} - H\vec{f}]^T[\vec{g} - H\vec{f}] = \epsilon,$$

where matrices H and L are block-circulant, and $\epsilon > 0$ is a fixed parameter.

   (a) Let $W = W_2 \otimes W_2$, where $W_2(k,n) = \frac{1}{\sqrt{2}}e^{\pi jkn}$ for $0 \leq k,n \leq 1$ and $\otimes$ is the Kronecker product. Given that $H = W\Lambda_H W^{-1}$ and $L = W\Lambda_L W^{-1}$, where

$$\Lambda_H = \begin{pmatrix} h_0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_3 \end{pmatrix} \text{ and } \Lambda_L = \begin{pmatrix} l_0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix}.$$

   where $h_i, l_i \in \mathbb{R}^+, 0 \leq i \leq 3$.
   Let $\vec{g} = \mathcal{S}(g)$, where g is a 2×2 image and $\mathcal{S}$ is the stacking operator.

      i. Show that H is block-circulant
      ii. Show that $W^{-1}\mathcal{S}(h) = 2\mathcal{S}(\hat{h})$ for any 2×2 image h.

   (b) As we have shown in the previous homework, the optimal solution $\vec{f} = \mathcal{S}(f)$ that solves the constrained least square problem satisfies $[\lambda H^T H + L^T L]\vec{f} = \lambda H^T \vec{g}$ for some parameter λ. Find $DFT(f)$ in term of $DFT(g)$, $h_i, l_i, 0 \leq i \leq 3$ and λ. You may assume $\lambda > 0$. Please show your answer with details.

2. Consider a periodically extended 4×4 image $I = (I(x,y))_{0 \leq x,y \leq 3}$ given by:

$$I = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & 0 & a \\ 1 & c & 0 & 0 \end{pmatrix}$$

Given that the discrete Laplacian ΔI of I is given by the formula:

$$\Delta I(x,y) = -4I(x,y) + I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) \text{ for } 0 \leq x,y \leq 3.$$

We perform the Laplacian masking on I to get a sharpen image I_{sharp}. Suppose I_{sharp} is given by

$$I_{sharp} = \begin{pmatrix} -2 & -2 & 4 & 3 \\ -1 & 0 & -2 & 5 \\ 0 & -1 & 1 & -6 \\ 4 & 4 & -2 & -1 \end{pmatrix}.$$

Find a, b and c. (**Hint**: You may want to use the formula of Laplacian masking in the spatial domain: $I_{sharp} = I - \Delta I$.)

3. Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \leq k,l \leq 3}$ given by:

$$I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b & 0 & c \end{pmatrix},$$

where a, b and c are real numbers.

We apply the 3×3 mean filter to I to obtain I^{mean}. Suppose $I^{mean}(0,1) = 1$, $I^{mean}(1,2) = 8/9$ and $I^{mean}(2,2) = 5/9$. Find a, b and c.

4. Consider the following periodically extended 4×4 image $f = (f(x,y))_{0 \leq x,y \leq 3}$:

$$f = \begin{pmatrix} 5 & 0 & 9 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Consider the ideal low-pass filter $H_{LP} = (H_{LP}(x,y))_{0 \leq x,y \leq 3}$ of radius 2. Explain with details why H_{LP} is given by:

$$H_{LP} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

(b) We apply the ideal low-pass filter of radius 2, H_{LP} to perform unsharp masking on f to get f_{sharp}. Find f_{sharp}. Please show all your steps.

5. Consider an image denoising model for $f : [a,b] \times [a,b] \to \mathbb{R}$:

$$E(f) = \int_a^b \int_a^b \left[(f(x,y) - g(x,y))^2 + 2K(x,y) \|\nabla f(x,y)\|^2 \right] dxdy$$

Assuming $f(x,y) = g(x,y) = 0$ for (x,y) on the boundary of $[a,b] \times [a,b]$. Show that when f minimizes $E(f)$, the f satisfies:

$$f(x,y) - g(x,y) - 2\nabla \cdot (K(x,y)\nabla f(x,y)) = 0 \text{ in } [a,b] \times [a,b]$$