Lecture 12:

Image denoising using energy minimization

Let g be a noisy image corrupted by additive noise n.

Then: $g(x,y) = \underbrace{f(x,y)}_{\text{Clean image}} + \underbrace{n(x,y)}_{\text{noise}}$

Recall: Laplacian masking : $g = f - \Delta f$ (Obtain a sharp (non-smooth) image from a smooth image)

Conversely, to get a smooth image f from a non-smooth image g, we can solve the PDE for f : $-\Delta f + f = g$

unknown (f), known (g)

We will show that solving the above equation is equivalent to minimizing something :

$$E(f) = \iint \left(f(x,y) - g(x,y)\right)^2 dx\,dy + \iint \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx\,dy$$

In the discrete case, the PDE can be approximated (discretized) to get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)]$$

for all (x,y) (Linear System)

Consider: $E_{discrete}(f) = \sum_{x=1}^{N}\sum_{y=1}^{N}(f(x,y) - g(x,y))^2 + \sum_{x=1}^{N}\sum_{y=1}^{N}\left[(f(x+1,y) - f(x,y))^2 + (f(x,y+1) - f(x,y))^2\right]$

Suppose f is a minimizer of $E_{discrete}$. Then, for each (x,y),

$E_{discrete}$ depends on $f(x,y)$ for each (x,y)

$$\frac{\partial E_{discrete}}{\partial f(x,y)} = 0.$$

$$= 2(f(x,y) - g(x,y)) + 2(f(x+1,y) - f(x,y))(-1) + 2(f(x,y+1) - f(x,y))(-1)$$
$$+ 2(f(x,y) - f(x-1,y)) + 2(f(x,y) - f(x,y-1))$$

By simplification, we get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)]$$

The continuous version of $E_{discrete}$ can be written as:

$$E(f) = \iint (f(x,y) - g(x,y))^2 + \iint \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]}_{|\nabla f|^2} dx\,dy$$

Remark:

- Solving $f = g + \Delta f$ is equivalent to energy minimization
- The first term in Ediscrete is called the fidelity term. Aim to find f that is close to g.
- The second term is called the regularization term. Aim to enhance Smoothness.
- $-\nabla f + f = g$ can also be solved in the frequency domain:

$$DFT(f) = DFT(g + \underbrace{\Delta f}_{p * f})$$

$$\therefore\ DFT(f)(u,v) = DFT(g)(u,v) + c\, DFT(p)(u,v)\, DFT(f)(u,v)$$

$$\Longleftrightarrow\ DFT(f)(u,v) = \left[\frac{1}{1 - c\, DFT(p)(u,v)}\right] DFT(g)(u,v)$$

$\downarrow$ inverse DFT

$f(x,y)$!!

2D integration by part formula

Let $f : [a,b]\times[a,b] \to \mathbb{R}$ and $g : [a,b]\times[a,b] \to \mathbb{R}$.

Assume $f(a,y) = f(b,y) = f(x,a) = f(x,b) = 0$.

$g(a,y) = g(b,y) = g(x,a) = g(x,b) = 0$.

Then: $$\int_a^b\int_a^b \nabla f(x,y)\cdot\nabla g(x,y)\,dx\,dy = -\int_a^b\int_a^b \Delta f(x,y)\, g(x,y)\,dx\,dy$$

<u>Proof</u>: $$\int_a^b\int_a^b \underbrace{\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}}_{\nabla f\cdot\nabla g}\,dx\,dy = -\int_a^b\int_a^b\left(\frac{\partial^2 f}{\partial x^2}\right) g\,dx\,dy + \cancelto{0}{\int_a^b \left(\frac{\partial f}{\partial x}\right) g\Big|_{x=a}^{x=b} dy}$$

$$-\int_a^b\int_a^b\left(\frac{\partial^2 f}{\partial y^2}\right) g\,dx\,dy + \cancelto{0}{\int_a^b \frac{\partial f}{\partial y}\, g\Big|_{y=a}^{y=b} dx}$$

$$= -\int_a^b\int_a^b \underbrace{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}_{\Delta f} g\,dx\,dy$$

Also,

$$\int_a^b\int_a^b \Big(k(x,y)\,\nabla f(x,y)\Big)\cdot \nabla g(x,y)\,dx\,dy = -\int_a^b\int_a^b \underbrace{\nabla\cdot}_{\text{divergence}}\Big(k(x,y)\,\nabla f(x,y)\Big)\,g(x,y)\,dx\,dy$$

where $k:[a,b]\times[a,b]\to\mathbb{R}$.

$$\nabla\cdot\begin{pmatrix}V_1\\ V_2\end{pmatrix} = \frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}$$

Proof:

$$\int_a^b\int_a^b \left[k(x,y)\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + k(x,y)\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right]dx\,dy$$

$$= -\int_a^b\int_a^b \frac{\partial}{\partial x}\left(k(x,y)\frac{\partial f}{\partial x}\right) g\,dx\,dy + \int_a^b \cancelto{0}{k(x,y)\frac{\partial f}{\partial x}\,g\Big|_{x=a}^{x=b}}\,dy$$

$$-\int_a^b\int_a^b \frac{\partial}{\partial y}\left(k(x,y)\frac{\partial f}{\partial y}\right) g\,dx\,dy + \int_a^b \cancelto{0}{k(x,y)\frac{\partial f}{\partial y}\,g\Big|_{y=a}^{y=b}}\,dx$$

$$= -\int_a^b\int_a^b \underbrace{\left[\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,y)\frac{\partial f}{\partial y}\right)\right]}_{\nabla\cdot\left(k(x,y)\,\nabla f\right)} g\,dx\,dy$$

Another useful fact:

If: $\int_\Omega T(x,y)\, v(x,y)\, dx\, dy = 0$ for all $v(x,y)$

then, we can conclude $T(x,y) = 0$ in Ω

Image denoising by solving PDE (derived from energy minimisation problem)

Consider the harmonic - L2 minimization model:

$$\text{minimize} \quad E(f) = \underbrace{\int_a^b \int_a^b}_{\text{image domain}} \Big(f(x,y) - \underbrace{g(x,y)}_{\text{Observed}} \Big)^2 \, dx\,dy + \int_a^b \int_a^b \underbrace{|\nabla f|^2}_{\text{Smoothness of } f} \, dx\,dy$$

(Look for (continuous) image f)

Assume that $f(x,y) = g(x,y) = 0$ on the boundary of $[a,b] \times [a,b]$.

Suppose f minimizes $E(f)$. Let $v : [a,b] \times [a,b] \to \mathbb{R}$ such that $v(x,y) = 0$ on the boundary of $[a,b] \times [a,b]$.

Consider $f^{\varepsilon} = f + \varepsilon v : [a,b] \times [a,b] \to \mathbb{R}$, which is another image with $f^{\varepsilon}(x,y) = 0$ on the boundary of $[a,b] \times [a,b]$.

$$f^{\varepsilon}(x,y) = \underset{0}{\underset{\shortparallel}{f(x,y)}} + \varepsilon \underset{0}{\underset{\shortparallel}{v(x,y)}} = 0 \quad \text{on } \partial\big([a,b] \times [a,b]\big).$$

Consider $S : \mathbb{R} \to \mathbb{R}$ defined by:

$$S(\varepsilon) \stackrel{\text{def}}{=} E(f^{\varepsilon}) = E(f + \varepsilon v).$$

Note that $S(0) = E(f) =$ minimum of E. Thus, S attains its minimum at $\varepsilon = 0$.

$$\therefore \quad \frac{dS}{d\varepsilon}(0) = 0.$$

Now,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\varepsilon) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(f + \varepsilon v) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\int_a^b \int_a^b \left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right)^2 dx\,dy + \int_a^b \int_a^b \left|\nabla(f + \varepsilon v)(x,y)\right|^2 dx\,dy \right)$$

$$|\nabla(f+\varepsilon v)|^2 = (\nabla f + \varepsilon \nabla v) \cdot (\nabla f + \varepsilon \nabla v) = \nabla f \cdot \nabla f + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 \nabla v \cdot \nabla v = |\nabla f|^2 + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 |\nabla v|^2$$

$$= \int_a^b \int_a^b 2\left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right) v(x,y) \Big|_{\varepsilon=0} dx\,dy$$

$$+ \int_a^b \int_a^b \left(2 \nabla f \cdot \nabla v + 2\varepsilon |\nabla v|^2\right)\Big|_{\varepsilon=0} dx\,dy$$

$$= \int_a^b \int_a^b 2\left(f(x,y) - g(x,y)\right) v(x,y)\, dx\,dy + \int_a^b \int_a^b 2 \nabla f(x,y) \cdot \nabla v(x,y)\, dx\,dy$$

$$\therefore \; S'(0) = 0 = 2\int_a^b\int_a^b \big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\, dy + 2\int_a^b\int_a^b \left(\frac{\partial f}{\partial x}(x,y)\, \frac{\partial v}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\, \frac{\partial v}{\partial y}(x,y) \right) dx\, dy$$

for all $v(x,y)$. — $(*)$

If we can formulate $(*)$ in the form:

$$\int_a^b\int_a^b T(x,y)\, v(x,y) = 0 \quad \text{for all } v(x,y),$$

then we can conclude that $T(x,y) = 0$ in $[a,b] \times [a,b]$.

Remark:
- First term is in the form $\int_a^b\int_a^b T(x,y)\, v(x,y)$
- Second term is NOT.

Need to reformulate the second term.

Strategy: integration by part.

Second term: $\int_a^b \int_a^b \nabla f(x,y) \nabla v(x,y) \, dx\, dy = -2 \int_a^b \int_a^b \Delta f(x,y) \, v(x,y) \, dx\, dy.$

All together, we have

$$0 = S'(0) = \int_a^b \int_a^b 2\big(f(x,y) - g(x,y)\big) v(x,y) \, dx\, dy - 2\int_a^b \int_a^b \Delta f(x,y) \, v(x,y) \, dx\, dy$$

$$\therefore \int_a^b \int_a^b \Big(2\big(f(x,y) - g(x,y)\big) - 2\Delta f(x,y)\Big) v(x,y) \, dx\, dy = 0 \text{ for all } v(x,y).$$

We conclude:

$$2\big(f(x,y) - g(x,y)\big) - 2\Delta f(x,y) = 0 \text{ for } (x,y) \in [a,b]\times[a,b]$$

or

$$f(x,y) - g(x,y) - \Delta f(x,y) = 0$$ (converse of Laplacian masking !!)

Example: Consider an image denoising model to find $f: \underbrace{[a,b]\times[a,b]}_{D} \to \mathbb{R}$ that minimizes:

$$E(f) = \int_a^b \int_a^b (f(x,y)-g(x,y))^2 \, dxdy + \int_a^b \int_a^b |\nabla f(x,y)|^4 \, dxdy.$$

Suppose f minimizes $E(f)$. Assume $f(x,y) = g(x,y) = 0$ for all $(x,y) \in \partial D$. Find a partial differential equation that f must satisfy.

Solution: Suppose f minimizes $E(f)$. For any $v: D \to \mathbb{R}$ such that $v(x,y) = 0$ on ∂D, we have:

$$\begin{cases} f^\varepsilon \overset{\text{def}}{=} f + \varepsilon v \text{ is an image with} \\ f^\varepsilon(x,y) = f(x,y) + \varepsilon v(x,y) = 0 \text{ on } \partial D. \end{cases}$$

Consider $S : \mathbb{R} \to \mathbb{R}$ where $S(\varepsilon) \overset{\text{def}}{=} E(f^{\varepsilon}) = E(f + \varepsilon v)$

Then, $S(0) = E(f) =$ minimum of E. Thus, S attains minimum at $\varepsilon = 0$.

$\therefore \left.\dfrac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0$ for all $v : D \to \mathbb{R}$

Now,

$$0 = \left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\int_D \big(f(x,y) + \varepsilon v(x,y) - g(x,y)\big)^2 \, dx\, dy + \int_D |\nabla(f+\varepsilon v)(x,y)|^4 \, dx\, dy \right)$$

$$\left(|\nabla f + \varepsilon \nabla v|^2\right)^2$$
$$\parallel$$
$$\big((\nabla f + \varepsilon \nabla v) \cdot (\nabla f + \varepsilon \nabla v)\big)^2$$
$$\parallel$$
$$\left(\nabla f \cdot \nabla f + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 \nabla v \cdot \nabla v\right)^2$$
$$\parallel$$
$$\left(|\nabla f|^2 + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 |\nabla v|^2\right)^2$$

$$\therefore \quad 0 = \frac{dS}{d\varepsilon}(0) = \int_D 2\left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right) v(x,y)\Big|_{\varepsilon=0} dxdy$$

$$+ \int_D 2\left(|\nabla f|^2 + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 |\nabla v|^2\right)\left(2\nabla f \cdot \nabla v + 2\varepsilon |\nabla v|^2\right)\Big|_{\varepsilon=0} dxdy$$

$$\Rightarrow \quad 0 = \int_D 2(f(x,y) - g(x,y))\, v(x,y)\, dxdy + \int_D 4\left(|\nabla f|^2\right) \nabla f \cdot \nabla v \, dxdy$$

$$= \int_D 2\left(f(x,y) - g(x,y)\right) v(x,y)\, dxdy - \int_D \left(4\, \nabla \cdot \left(|\nabla f|^2\, \nabla f\right)(x,y)\right) v(x,y)\, dxdy$$

All together, we have:

$$0 = \int_D \left(2\left(f(x,y) - g(x,y)\right) - 4\, \nabla \cdot \left(|\nabla f(x,y)|^2\, \nabla f(x,y) \right) \right) v(x,y)\, dx\, dy$$

for all $v(x,y)$.

We can conclude that:

$$f(x,y) - g(x,y) - 4\, \nabla \cdot \left(|\nabla f(x,y)|^2\, \nabla f(x,y) \right) = 0 \text{ in } D.$$

(Partial differential equation)

Total variation (TV) denoising (ROF)

Invented by: Rudin, Osher, Fatemi

Motivation: Previous model: $f = g + \Delta f$. Solve for f from noisy g.

Disadvantage: smooth out edge.

Modification: $f = g + \nabla \cdot (K \nabla f)$ K is small on edges!!

Goal: Given a noisy image $g(x,y)$, we look for $f(x,y)$ that solves:

$$f = g + \lambda \frac{\partial}{\partial x}\left(\frac{1}{|\nabla f|(x,y)} \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{|\nabla f|(x,y)} \frac{\partial f}{\partial y}\right) \qquad (*)$$

Remark: Problem arises if $|\nabla f(x,y)| = 0$. Take care of it later.

We'll show that $(*)$ must be satisfied by a minimizer of:

$$J(f) = \frac{1}{2}\int_\Omega (f(x,y) - g(x,y))^2 + \underbrace{\lambda}_{\text{constant parameter} > 0} \int_\Omega |\nabla f(x,y)| \, dx\, dy$$

Same idea: Let $S(\varepsilon) := E(f + \varepsilon v)$

$$= \frac{1}{2}\int_\Omega (f + \varepsilon v - g)^2 + \lambda \int_\Omega \underbrace{|\nabla f + \varepsilon \nabla v|}_{\sqrt{(\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)}}$$

$$\frac{d}{d\varepsilon} S(\varepsilon) = \left[\int_\Omega (f + \varepsilon v - g)\, v + \lambda \int_\Omega \frac{\nabla f \cdot \nabla v + 2\varepsilon\, \nabla v \cdot \nabla v}{\sqrt{(\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)}} \right]$$

If f is a minimizer, $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon = 0} S(\varepsilon) = 0$ for all v.

$$\therefore\ S'(0) = 0 = \int_\Omega (f - g)\, v + \lambda \int_\Omega \frac{\nabla f \cdot \nabla v}{|\nabla f|}$$

$$= \int_\Omega (f - g)\, v - \lambda \int_\Omega \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right) v + \lambda \int_{\partial\Omega} \left(\frac{\nabla f}{|\nabla f|} \cdot \vec{n}\right) v$$

$$= \int_\Omega \left[(f - g) - \lambda\, \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right)\right] v + \lambda \int_{\partial\Omega} \left(\frac{\nabla f}{|\nabla f|} \cdot \vec{n}\right) v \quad \text{for all } v$$

We conclude: $(f - g) - \lambda\, \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right) = 0$!!

In the discrete case,

$$J(f) = \frac{1}{2}\sum_{x=1}^{N}\sum_{y=1}^{N}(f(x,y)-g(x,y))^2+\lambda\sum_{x=1}^{N}\sum_{y=1}^{N}\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}$$

J can be regarded as a multi-variable function depending on :

$f(1,1), f(1,2), \ldots, f(1,N), f(2,1), \ldots, f(2,N), \ldots, f(N,N)$.

If f is a minimizer, then $\frac{\partial J}{\partial f(x,y)} = 0$ for all (x,y).

$$\begin{aligned}\frac{\partial J}{\partial f(x,y)} &= (f(x,y)-g(x,y)) + \lambda\frac{2(f(x+1,y)-f(x,y))(-1)+2(f(x,y+1)-f(x,y))(-1)}{2\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}} \\ &+ \lambda\frac{2(f(x,y)-f(x-1,y))}{2\sqrt{(f(x,y)-f(x-1,y))^2+(f(x-1,y+1)-f(x-1,y))^2}} \\ &+ \lambda\frac{2(f(x,y)-f(x,y-1))}{2\sqrt{(f(x+1,y-1)-f(x,y-1))^2+(f(x,y)-f(x,y-1))^2}} = 0\end{aligned}$$

By simplification:

$$f(x,y) - g(x,y) = \lambda \left\{ \frac{f(x+1,y) - f(x,y)}{\sqrt{(f(x+1,y) - f(x,y))^2 + (f(x,y+1) - f(x,y))^2}} - \frac{f(x,y) - f(x-1,y)}{\sqrt{(f(x,y) - f(x-1,y))^2 + (f(x-1,y+1) - f(x-1,y))^2}} \right\}$$

$$+ \lambda \left\{ \frac{f(x,y+1) - f(x,y)}{\sqrt{(f(x+1,y) - f(x,y))^2 + (f(x,y+1) - f(x,y))^2}} - \frac{f(x,y) - f(x,y-1)}{\sqrt{(f(x+1,y-1) - f(x,y-1))^2 + (f(x,y) - f(x,y-1))^2}} \right\}$$

Discretization of $f - g = \lambda \, \nabla \cdot \left(\frac{\nabla f}{|\nabla f|} \right)$

Gradient descent algorithm

Let $f \ \mathbb{R}^n \to \mathbb{R}$ We want to find a sequence $\vec{x}_0 \in \mathbb{R}^n, \vec{x}_1 \in \mathbb{R}^n, \quad , \vec{x}_n \in \mathbb{R}^n,$

such that $f(\vec{x}_0) \geq f(\vec{x}_1) \geq \quad \geq f(\vec{x}_n) \geq f(\vec{x}_{n+1}) \geq$

So, $\vec{x}_0, \vec{x}_1, \quad , \vec{x}_n,$ iteratively minimizes $f(\vec{x})$

Given $\vec{x}_0$, we want to find $\vec{x}_1 = \vec{x}_0 + t\vec{v}$ $(t > 0, \vec{v} \in \mathbb{R}^n)$ such that

$$f(\vec{x}_1) \leq f(\vec{x}_0)$$

Note that $f(\vec{x}_1) = f(\vec{x}_0 + t\vec{v}) \approx f(\vec{x}_0) + t \nabla f(\vec{x}_0) \vec{v} + \frac{t^2}{2} \vec{v}^T f''(\vec{x}_0) \vec{v} +$

(negluble)

Choose $\vec{v} = -\nabla f(\vec{x}_0)$ Then

$$f(\vec{x}_1) \approx f(\vec{x}_0) - t |\nabla f(\vec{x}_0)|^2 \leq f(\vec{x}_0)$$

Similarly, given $\vec{x}_n$, choose $\vec{v} = -\nabla f(\vec{x}_n)$ Let $\vec{x}_{n+1} = \vec{x}_n + t\vec{v} = \vec{x}_n + t \nabla f(\vec{x}_n)$

Then for small $t > 0$, we have

$$f(\vec{x}_{n+1}) \approx f(\vec{x}_n) - t |\nabla f(\vec{x}_n)|^2 \leq f(\vec{x}_n)$$

Therefore, we have an iterative scheme

$$\vec{x}_{n+1} = \vec{x}_n + t\,\vec{v}_n \;, \text{ where } \vec{v}_n = -\nabla f(\vec{x}_n)$$

$t > 0$ is small, called the time step

$\vec{v}_n \in \mathbb{R}^n$ is called the descent direction at n^{th} iteration

How to minimise $J(f)$

We consider the problem of finding f that minimizes $J(f)$.

In the discrete case, J depends on $f(x,y)$ for $x = 1, 2, \ldots, N$, $y = 1, 2, \ldots, N$.

Our goal is to find a sequence of images

$f^0, f^1, f^2, \quad, f^n, f^{n+1},$ such that $J(f_0) \geq J(f_1) \geq \quad \geq J(f_n) \geq J(f_{n+1}) \geq$

Define $$\nabla J(f^n) = \begin{pmatrix} \frac{\partial J}{\partial f^n(1,1)} \\ \frac{\partial J}{\partial f^n(2,1)} \\ \vdots \\ \frac{\partial J}{\partial f^n(N,1)} \\ \vdots \\ \frac{\partial J}{\partial f^n(1,N)} \\ \vdots \\ \frac{\partial J}{\partial f^n(N,N)} \end{pmatrix}$$

Given $\overrightarrow{f^n}$, define $\overrightarrow{f^{n+1}} = \overrightarrow{f^n} + \Delta t \vec{v}_n$

where $\vec{v}_n = -\nabla J(\overrightarrow{f^n})$

Here, $\overrightarrow{f^n}$ is the vectorized image of f^n

Then

$$J(\overrightarrow{f^{n+1}}) = J(\overrightarrow{f^n} + \Delta t \vec{v}_n) \approx J(\overrightarrow{f^n}) + \Delta t \nabla J(f^n)\, \vec{v}_n = J(\overrightarrow{f^n}) - \Delta t |\nabla J(f^n)|^2 \leq J(\overrightarrow{f^n})$$

In the discrete case,

$$\frac{\overrightarrow{f^{n+1}} - \overrightarrow{f^n}}{\Delta t} = -\nabla J(f^n)$$

(Gradient descent algorithm)

For the ROF model:

$$\begin{aligned}
&\frac{f^{n+1}(x,y) - f^n(x,y)}{\Delta t} \\
&= -(f^n(x,y) - g(x,y)) + \lambda \frac{f^n(x+1,y) - f^n(x,y)}{\sqrt{(f^n(x+1,y) - f^n(x,y))^2 + (f^n(x,y+1) - f^n(x,y))^2}} \\
&- \lambda \frac{f^n(x,y) - f^n(x-1,y)}{\sqrt{(f^n(x,y) - f^n(x-1,y))^2 + (f^n(x-1,y+1) - f^n(x-1,y))^2}} \\
&+ \lambda \frac{f^n(x,y+1) - f^n(x,y)}{\sqrt{(f^n(x+1,y) - f^n(x,y))^2 + (f^n(x,y+1) - f^n(x,y))^2}} \\
&- \lambda \frac{f^n(x,y) - f^n(x,y-1)}{\sqrt{(f^n(x+1,y-1) - f^n(x,y-1))^2 + (f^n(x,y) - f^n(x,y-1))^2}}
\end{aligned}$$

Discretization of ∇J

(Gradient descent algorithm for ROF)