A-priori estimate: $\nearrow$ $-\Delta\phi_a = a,\ \frac{\partial\phi_a}{\partial n}=0$,

If $\iint_{\Omega\times\mathbb{R}^3} f\sqrt{\mu}\,dx\,dv=0$, and $\int_{\gamma_-} r\sqrt{\mu}\,d\gamma=0$.

$\Rightarrow \|Pf\|_2^2 \lesssim \|(I-P)f\|_\nu^2 + \|g\|_2^2 + |(1-P_\gamma)f|_{2,+}^2 + |r|_2^2$.

different: ~~decompose~~ $f|_{\gamma}= P_\gamma f + r$,

extra contribution of r.

~~A~~ L^2 ✓, L^∞ ✓ for a-priori estimate.

Well-posedness? No Gronwall for the iteration sequence.

Penalized equation:

$$\varepsilon f + v\cdot\nabla_x f + Lf = g,\quad f|_{\gamma_-} = P_\gamma f + r.$$

Start from. $\varepsilon f^{l+1} + v\cdot\nabla_x f^{l+1} + \nu f^{l+1} = g$. $f^{l+1}|_{\gamma_-} = (1-\frac{1}{n}) P_\gamma f^{l} + r$.

$$\varepsilon\|f^{l+1}\|_2^2 + \int |f^{l+1}|_{2,+}^2 + \|f^{l+1}\|_\nu^2 \le (1-\tfrac{1}{n})^2 \left[\|f^l\|_\nu + |f^l|_{2,+}^2\right] + \cdots r \cdots g.$$

difference of $f^{l+1}-f^l \Rightarrow$ Cauchy sequence.

$\exists$ unique soln to

$$\varepsilon f + v\cdot\nabla_x f + \nu f = g,\quad f|_{\gamma_-} = (1-\tfrac{1}{n}) P_\gamma f + r.$$

L^∞ Cauchy sequence: ~~start~~ follows from L^2.

$$\varepsilon f + v\cdot\nabla_x f + \nu f - \lambda K f = g.$$

$$f|_{\gamma_-} = (1-\tfrac{1}{n}) P_\gamma f + r.$$

A-priori estimate: $\|f\|_{L^2} \lesssim_\varepsilon \|g\| + \|r\|$

$\|wf\|_{L^\infty} \lesssim_\varepsilon \|g\| + \|r\|$.

Fix point argument with small λ.

$\varepsilon f + v\cdot\nabla_x f + \nu f = \lambda K\tilde{f} + g$, well-posed.

$X = \{\|wf\| < \infty\}$, no contribution of Kf

$\|wf\|_\infty \le \|\frac{wg}{\langle v\rangle}\|_\infty + \lambda\|w\tilde{f}\|_\infty < \infty$ ✓.

Given $\tilde{f}_1, \tilde{f}_2$ and f_1, f_2.

$\|w(f_1-f_2)\|_\infty \le \lambda\|w(K\tilde{f}_1 - K\tilde{f}_2)\|_\infty \le \lambda\|w(\tilde{f}_1-\tilde{f}_2)\|_\infty$.

$\Rightarrow \exists!\ \varepsilon f + v\cdot\nabla_x f + \nu f - \lambda K f = g$,

$f|_{\gamma_-} = (1-\frac{1}{n}) P_\gamma f + r$ for small λ.

Next: $\varepsilon f + v\cdot\nabla_x f + \nu f - \lambda K f = \lambda K\tilde{f} + g$.

Already know from previous step, $\|wf\|_\infty < \infty$.

A-priori estimate can be applied

$\Rightarrow$ fix point thm $\Rightarrow \exists!$ soln.

$\lambda \to 2\lambda \to 3\lambda \to \cdots \to 1$.

$\exists!\ \varepsilon f + v\cdot\nabla_x f + Lf = g$, $f|_{\gamma_-} = (1-\frac{1}{n}) P_\gamma f + r$.

$\|f\|_{L^2} \lesssim_{\varepsilon} \|g\| + \|r\|$ uniform in n

$\Rightarrow \|wf\|_{\infty} \lesssim \|g\| + \|wr\|$ uniform in n.

$$\|f^{n_1} - f^{n_2}\|_2^2 \le \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(|f^{n_1}|_{2,+}^2 + |f^{n_2}|_{2,+}^2\right)$$

$$\le \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(\|wf^{n_1}\|_{\infty} + \|wf^{n_2}\|_{\infty}\right) \to 0.$$

$\Rightarrow$ unique soln to $\varepsilon f + v\cdot\nabla_x f + Lf = g$, $f|_{\gamma_-} = P_\gamma f + r$.

For ε: apply $L^2-\infty$ hypercoercivity:

Need to ensure $\int f\sqrt{\mu} = 0$, $\boxed{\text{need } \int_{\gamma_-} r\sqrt{\mu} = 0.}$

$\Rightarrow$ uniform in ε estimate

$\square$

Finally: $\exists$ unique $v\cdot\nabla_x f + Lf = g$, $f|_{\gamma_-} = P_\gamma f + r$.

Dynamical stability,

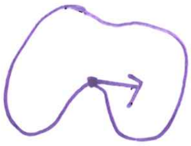
~~$\partial_t f + v\cdot\nabla_x f + Lf = \Gamma(f,f)$, $v\cdot\nabla_x f_s + Lf_s = \Gamma(f_s, f_s)$~~

$\partial_t F + v\cdot\nabla_x F = Q(F,F)$, $\quad v\cdot\nabla_x F_s = Q(F_s, F_s)$.

Perturbation around steady soln:

$F = F_s + \sqrt{\mu} f$, similar to initial value problem, with extra term involving r & F_s. $\square$

Regularity of Boltzmann eqn.



singularity propagates into Ω.

Assume domain is convex.

$$\Omega = \{x\in\mathbb{R}^3 : \xi(x)<0\}, \quad \partial\Omega = \{x\in\mathbb{R}^3 : \xi(x)=0\}.$$

Domain is strictly convex:

$$\sum_{i,j} \partial_{ij}\xi(x)\, z_i z_j \geq C_\xi |z|^2 \text{ for all } z\in\mathbb{R}^3.$$

$$n(x) = \frac{\nabla\xi(x)}{|\nabla\xi(x)|}.$$

$$f(t,x,v) = e^{-\nu t_b} f(t_1, x_b, v)$$

$\nabla_x t_b$ has singularity:

$$\xi(x_b) = \xi(x-t_b v) = 0, \quad \nabla_x[\xi(x_b)] = \nabla_x[\xi(x-t_b v)].$$

$$= \nabla_x \xi(x-t_b v)\,[I - \nabla_x t_b \otimes v]$$

$$= \nabla_x \xi(x-t_b) - \nabla_x t_b\, \nabla_x \xi(x-t_b v)\cdot v = 0$$

$$\Rightarrow \nabla_x t_b = \frac{\nabla_x \xi(x_b)}{\nabla_x \xi(x_b)\cdot v} = \frac{n(x_b)}{n(x_b)\cdot v} \sim \frac{1}{n(x_b)\cdot v}$$

Start from $\partial_t f + v\cdot\nabla_x f + \nu f = H$

$f|_{\gamma_-} = g$.

$$f(t,x,v) = 1_{t_1>0}\, e^{-\nu t_b} f(t_1, x_b, v) + 1_{t_1\leq 0}\, e^{-\nu t} f(0, x-tv, v) + \int_{\max\{0,t_1\}}^{t} e^{-\nu(t-s)} H(s, x-(t-s)v, v)\, ds.$$

Issue of continuity:

Need compatibility condition:

$$f_0|_{\gamma^-} = g(0)$$

$\Rightarrow$ Piece-wise continuous $f(t,x,v)$ across $\{t = t_b\}$.

$\Rightarrow$ weak derivative given by:

$$\left.\begin{aligned}\nabla_x f(t,x,v) = {} & 1_{t_1\geq 0}\cdot \nabla_x\left[e^{-\nu t_b} f(t_1, x_b, v)\right] \\ & + 1_{t_1<0}\, \nabla_x\left[e^{-\nu t} f(0, x-tv, v)\right] \\ & + \nabla_x\left[\int_{\max\{0,t_1\}}^{t} e^{-\nu(t-s)} H(s, x-(t-s)v, v)\right]\end{aligned}\right\} (*)$$

In fact, for test function ϕ.

$$\int \nabla_x\phi\, f = \int_{t_1\geq 0} \nabla_x\phi\, f + \int_{t_1<0} \nabla_x\phi\, f$$

$$= -\int_{t_1\geq 0} \phi\, \nabla_x f - \int_{t_1<0} \phi\, \nabla_x f$$

$$+ \int_{t_1=0} \underbrace{\left(\int \phi \left(e^{-\nu t} g(0, x_1, v) - e^{-\nu t} f(0, x_1, v)\right) n \right)}_{=0}$$

$\downarrow$

$\nabla t_b \cdot \vec{\tau}_{\text{tangent}} = 0$

thus n is same direction of $n(x_b)$.

Thus $(*)$ is the weak derivative.

$\nabla_x f(t,x,v)$ has singularity $\nabla_x f(t,x,v) \sim \frac{1}{n(x_b)\cdot v}$.

Natural choice: try to bound $|(n(x_b)\cdot v)\nabla_x f(t,x,v)|$.

does not work with C^1-norm, since linear operator:

$$\int_0^t e^{-v(t-s)} \int_{\mathbb{R}^3} \nabla_x f(s, x-(t-s)v, u)\, k(v,u)\, du.$$

$\int_{\mathbb{R}^3} \frac{1}{n(x)\cdot u}$ not locally integrable.

Kinetic weight: $\alpha(x,v) = |v\cdot\nabla\xi(x)|^2 - 2\{v\cdot\nabla^2\xi(x)\cdot v\}\,\xi(x)$.

Property ① : on the bdr: $\sqrt{\alpha(x,v)} = |v\cdot\nabla\xi(x)| \sim n(x)\cdot v$.

② convexity: $v\cdot\nabla^2\xi(x)\cdot v \gtrsim |v|^2$, $\xi(x)\leq 0$, $\alpha(x,v)\geq 0$

Goal: bound $\nabla_x f$ with extra weight: $\|\alpha\nabla_x f\|_{L^\infty}$.

Velocity Lemma:

$$e^{-C|v|s}\alpha(x-sv,v) \lesssim \alpha(x,v) \lesssim e^{+C|v|s}\alpha(x-sv,v)$$

$|v|s \leq \mathrm{diam}(\Omega)$: means $\alpha(x-sv,v) \sim \alpha(x,v)$, and

$\alpha(x,v) \sim \alpha(x_b(x,v),v) \sim n(x_b)\cdot v$: singularity.

Proof: $v\cdot\nabla_x\alpha = 2(v\cdot\nabla\xi(x))[v\cdot\nabla^2\xi\cdot v] - 2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v]$

$- 2v\{v\cdot\nabla^3\xi(x)\cdot v\}\,\xi(x)$

$= -2v\{v\cdot\nabla^3\xi(x)\cdot v\}\,\xi(x) \sim |v|^3|\xi(x)| \lesssim |v|\alpha$.

$$\Rightarrow \quad -C|v|\alpha(x,v) \le v\cdot\nabla_x \alpha(x,v) \le C|v|\alpha(x,v)$$

$$\frac{d}{ds}\alpha(x-sv,v) = -v\cdot\nabla_x\alpha(x-sv,v)$$

$$-C|v|\alpha(x-sv,v) \le \frac{d}{ds}\alpha(x-sv,v) \le C|v|\alpha(x-sv,v)$$

Gronwall

$\square$

For the same reason,

$[\partial_t + v\cdot\nabla_x]\,\alpha(x,v) \lesssim \langle v\rangle\,\alpha(x,v)$; In order to control extra term,

Need to introduce $e^{-w\langle v\rangle t}\alpha(x,v)$.

$$\{\partial_t + v\cdot\nabla_x\}[e^{-w\langle v\rangle t}\alpha(x,v)] = -w\langle v\rangle e^{-w\langle v\rangle t}\alpha(x,v)$$

$$+ e^{-w\langle v\rangle t}|v|\,\alpha(x,v) \le (-w+C)\langle v\rangle e^{-w\langle v\rangle t}\alpha(x,v)$$

$\Rightarrow$ Strong growing factor in t, can not expect uniform-in-time estimate.

Another way to understand: $\frac{1}{C} \le \left|\frac{\alpha(x_1,v_1)}{\alpha(x_2,v_1)}\right| \le C$, in the $\alpha\Sigma_k$ have C^k. ~~have to~~.

$\Rightarrow$ Study Local-in-time regularity.

Come back to: $\partial_t f + v\cdot\nabla_x f + \nu f = \Gamma(f,f)$.

$$f = 1_{t_1>0}\,\nabla_x[e^{-\nu t_b} f(t_1,x_b,v)] \quad (1)$$

$$+\, 1_{t_1<0}\,\nabla_x[e^{-\nu t} f(0,x-tv,v)] \quad (2)$$

$$+\, \nabla_x\Big[\int_{\max\{0,t_1\}}^{t} e^{-\nu(t-s)}\,\Gamma(f,f)(s,x-(t-s)v,v)\Big] \quad (3)$$

$e^{\nu t_b}$ (1.1) (1.2)

$(1) = \nabla_x t_1 \partial_t f + e^{-\nu t_b} \nabla_{x_b} f(t_1, x_b, v) \cdot \nabla_x x_b + \nabla_x t_b \, e^{-\nu t_b} f$ ✓

~~$\partial_t f$~~

(1.1): $\partial_t f + v \cdot \nabla_x f + \nu f = \Gamma(f, f)$, $f|_{\gamma} = P_\gamma f$,

Take t-derivative: $(\partial_t + v \cdot \nabla_x + \nu)(\partial_t f) = \partial_t(\Gamma(f, f))$.

$f|_{\gamma^-} = P_\gamma \partial_t f$, just apply argument

in local well-posedness with $\|wf\|_\infty < \infty$.

(1.2). $\nabla_x x_b \sim \frac{1}{n(x_b) \cdot v} \sim \frac{1}{\alpha(x, v)}$ ✓

$\nabla_{x_b} f(t_1, x_b, v)$: tangential derivative at the boundary.

$f(x)|_{\gamma^-} = \sqrt{\mu(v)} \int f(x, u) \sqrt{\mu(u)} (n(x) \cdot u) \, du$.

can take tangential derivative τ_i

: $\partial_{\tau_i} f(x)|_{\gamma^-} = \sqrt{\mu(v)} \int \partial_{\tau_i} f(x, u) \sqrt{\mu(u)} (n(x) \cdot u) \, du$.

Iterate again $\Rightarrow C^k \, d\Sigma_k$

① k is still finite under diffuse bc.

② C^k large is fine for the integral condition.

③ for Duhamel principle, always use small t.

Difficulty: how to control $\nabla_x(\Gamma(f, f)) = \Gamma(\nabla_x f, f) + \Gamma(f, \nabla_x f)$.

$\int_{t_2}^{t_1} \Gamma(\nabla_x f, f) \le \|wf\|_\infty \cdot \Gamma(\nabla_x f, w^{-1}, 1)$.