~~Example~~

Mismatch between boundaries

Steady Boltzmann equation

$$\begin{cases} v\cdot\nabla_x F = \frac{1}{\varepsilon} Q(F,F) \\ F(x,v)\big|_{\gamma^-} = M_w(x,v)\int_{n\cdot u>0} F(x,u)\,|n(x)\cdot u|\,du . \end{cases}$$

$M_w(x,v) = \frac{1}{2\pi (T_w(x))^2} \exp\left(\frac{-|v|^2}{2T_w(x)}\right)$. $\| T_w(x) - 1 \| \ll 1 \to$ not $O(\varepsilon)$.

$$F(x,v) = \mu + \sqrt{\mu}\,(\varepsilon f_1 + \varepsilon^2 f_2 + \cdots) .$$

μ can not be global Maxwellian, since $\varepsilon \to 0$, μ does not satisfy boundary condition.

Then. $\mu(x,v)$ denotes a local Maxwellian

Comparing order:

$O(\frac{1}{\varepsilon})$: $Q(\mu,\mu) = 0 \Rightarrow \mu = \frac{\rho(x)}{(2\pi T(x))^{\frac{3}{2}}} \exp\left(\frac{-|v|^2}{2T(x)}\right)$. $T(x) = T_w(x)$ at wall.

$O(\varepsilon^0)$: $v\cdot\nabla_x \mu - 2Q(\mu, \sqrt{\mu} f_1) - 2Q(\sqrt{\mu} f_1, \mu) = 0$

$O(\varepsilon)$: $v\cdot\nabla_x(\sqrt{\mu} f_1) - 2Q(\mu, \sqrt{\mu} f_2) - 2Q(\sqrt{\mu} f_2, \mu) = \cancel{\Gamma(f_1, f_1)}$ $Q(\sqrt{\mu} f_1, \sqrt{\mu} f_1)$.

$O(\varepsilon^2)$: $\mu^{\frac{1}{2}}\, v\cdot\nabla_x(\mu^{\frac{1}{2}} f_2) \perp \operatorname{Ker}(L)$.

$O(\varepsilon^0)$: $\mu^{-\frac{1}{2}}(v\cdot\nabla_x\mu)+\mathcal{L}(f_1)=0.$

$\mu^{-\frac{1}{2}}(v\cdot\nabla_x\mu)\perp \mathrm{Ker}(\mathcal{L}) \Rightarrow \quad \nabla_x(\rho T)=0, \quad \frac{\nabla_x\rho}{\rho}=\frac{-\nabla_x T}{T}$

$\Rightarrow \quad v\cdot\nabla_x\mu=\mu(v\cdot\nabla_x T)\frac{|v|^2-5T}{2T^2}, \quad \cancel{\mathcal{L}(f_1)=}$

$$\Rightarrow f_1 = -\bar{A}\cdot\frac{\nabla_x T}{2T^2}+\sqrt{\mu}\left(\frac{\rho_1}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T)}{2T^2}\right).$$

Boundary Matching:

$$\mu|_{n\cdot v<0}=M_w\int_{n\cdot u>0}\mu|n\cdot u|\,du \cdot \surd .$$

Next order:

$$\mu^{\frac{1}{2}}f_1 \neq M_w\int_{n\cdot v'>0}\sqrt{\mu}f_1(n\cdot u)\,du+O(\varepsilon).$$

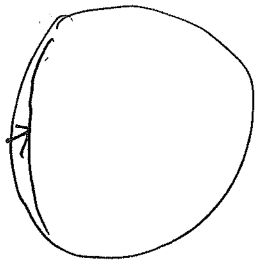
Need correction f_1^B to have

$$\mu^{\frac{1}{2}}(f_1+f_1^B)=M_w\int_{n\cdot v'>0}\sqrt{\mu}(f_1+f_1^B)(n\cdot u)\,du+O(\varepsilon).$$

At boundary $f_1^B|_{\gamma^-} = \bar{A}\cdot\frac{\nabla_x T}{2T^2}\phi-\left(\frac{\rho^B}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T_w)}{2T^2}\right)$

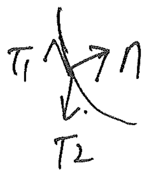
need

Interior: want $f_1^B\to$ fluid.



transition in the normal direction.

Linear equation: $v\cdot\nabla_x f + \frac{\mathcal{L}f}{\varepsilon} = \cancel{Q} \, \Gamma(f,f)$.



and set. $\eta = \frac{n}{\varepsilon} \Rightarrow \frac{\partial}{\partial n} = \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}$.

$v\cdot\nabla_x = \frac{1}{\varepsilon} v_\eta \frac{\partial}{\partial \eta} + O(1) \Rightarrow$

Order $O(\varepsilon^{-1})$: $v_\eta \frac{\partial}{\partial \eta} f + \mathcal{L} f = 0$ (Milne's problem

Milne problem.

$$\xi_1 \partial_x f + \mathcal{L} f = 0.$$

Goal: solve the Boltzmann equation for $0 \le x < \infty$, with incoming particle at $x=0$ & boundedness condition at $x=\infty$.

$$\Rightarrow \begin{cases} \xi_1 \partial_x f + \mathcal{L} f = 0. \\ f = g, \quad x=0, \quad \xi_1 > 0. \\ \int \xi_1 M^{1/2} f \, d\xi = m_f. \end{cases}$$

Assume $\int_{\xi_1>0} (1+|\xi|) g^2 \, d\xi = K_g < \infty$, $(*)$: $D = \{ f : (1+|\xi|)^{\frac{1}{2}} f \in L^\infty(\mathbb{R}_x^+, L^2(\mathbb{R}^3_\xi))$, $f_x \in L^2_{loc}(\mathbb{R}_x^+, L^2(\mathbb{R}^3_\xi)) \}$.

Theorem: for any m_f and g satisfying $(*)$,

$\exists !\, f \in D$.

Decompose $f = w + q$, ~~the~~ where $w \in N^\perp(\mathcal{L})$, $q \in N(\mathcal{L})$.

Then
$$\frac{d}{dx} \int \xi_1 \cancel{\,} M^{\frac{1}{2}} f \, d\xi = 0,$$
$$\frac{d}{dx} \int \xi_1 \xi_i M^{\frac{1}{2}} f \, d\xi = 0.$$
$$\frac{d}{dx} \int \xi_1 \xi^2 M^{1/2} f \, d\xi = 0, \quad \text{and}$$

$$\lim_{x \to \infty} q = (a_\infty + m_f \xi_1 + b_{2\infty} \xi_2 + b_{3\infty} \xi_3 + c_\infty \xi^2) M^{1/2}.$$

with $|a_\infty| + |b_{2,\infty}| + |b_{3\infty}| + |c_\infty| < k$

$$\int (1+|\xi|)\, w^2 d\xi + |a-a_\infty|^2 + |b_2 - b_{2\infty}|^2 + |b_3 - b_{3\infty}|^2 + |c-c_\infty|^2.$$
$$\le k(\nu_0-\gamma)^{-2} e^{-2\gamma x},$$

and for any $\delta>0$, $0\le\gamma<\nu_0$

$$\int_\delta^\infty \int (1+|\xi|)\, e^{2\gamma x} f_x^2 \, d\xi\, dx \le k_\delta (\nu_0-\gamma)^{-3}$$

Orthogonality and asymptotic properties:

Suppose $f\in D$ satisfies the problem,

$(\xi_1 \partial_x f + Lf = 0)\cdot \psi_i$

$\Rightarrow$. $\frac{d}{dx}\int \xi_1 \phi \cdot \sqrt{\mu}\, f\, d\xi = 0$, $\Rightarrow$ $b_{1,\infty} = m_f$.

$$\frac{d}{dx}\int \xi_1\xi_i \sqrt{\mu} f d\xi = \frac{d}{dx}\int \xi_1 \xi^2 \sqrt{\mu} f\, d\xi.$$

Preparation for asymptotic properties:

Denote $\tilde{f} = f - m_f \xi_1 M^{\frac{1}{2}} = w + \tilde{q}$.

$\tilde{q} = (a + b_2\xi_2 + b_3\xi_3 + c\xi^2) M^{1/2}$,

then $\tilde{f}\in D$ solves

$(*)$. $\xi_1 \frac{d}{dx}\tilde{f} + \mathcal{L}\tilde{f} = 0$, $\quad x>0$.

$$\tilde{f} = \tilde{g} = g - m_f \xi_1 M^{\frac{1}{2}}, \quad x=0, \ \xi_1>0.$$

$$\int \xi_1 \tilde{q}^2 d\xi = \int \xi_1 \psi_i \tilde{q}\, d\xi = 0, \quad i \neq 1,$$

$$\Rightarrow \frac{d}{dx}\int \xi_1 \psi_i w\, d\xi = 0. \quad (**)$$

Energy estimate to $(*)$:

$$\frac{d}{dx}\int \xi_1 \tilde{f}^2 d\xi + \nu_0 \int (1+|\xi|)w^2 d\xi \leq 0,$$

Since. $\tilde{f} \in D$, then.

$$\int_0^\infty \int (1+|\xi|)w^2 d\xi\, dx < \infty.$$

Then $\int \xi_1 \psi_i w\, d\xi \leq C\int (1+|\xi|)\, w^2 d\xi \in L^1(\mathbb{R}_x^+)$.

$(**) \Rightarrow$. $\int \xi_1 \tilde{q} w\, d\xi = \underbrace{\int \xi_1 \psi_i w\, d\xi}_{\in L^1(\mathbb{R}_x^+)} = 0, \quad i \neq 1.$

$\Rightarrow \int \xi_1 \tilde{f}^2 d\xi = \int \xi_1 w^2 d\xi$. $\left(\tilde{f}^2 = w^2 + 2w\tilde{q} + \tilde{q}^2. \right)$.

Integrated decay of w:

energy estimate can be rewritten as.

$$\frac{d}{dx}\int \xi_1 w^2 d\xi + \nu_0 \int (1+|\xi|) w\, d\xi \leq 0.$$

$\Rightarrow \int \xi_1 w^2 d\xi$ is decreasing, since it is in $L^1(\mathbb{R}_x^+)$,

$$\int \xi_1 w(x)^2 d\xi \to 0 \quad \text{as } x \to \infty.$$

$$0 \le \int \xi_1 w(x)^2 d\xi \le \int \xi_1 w(x=0)^2 d\xi = \int \xi_1 \tilde{f}(x=0)^2 d\xi.$$

↳ positive.

$$\le \int_{\xi_1 > 0} \xi_1 \tilde{g}^2 d\xi \le K_g + m_f^2.$$

Multiplying $\xi_1 \frac{\partial}{\partial x} \tilde{f} + \mathcal{L}\tilde{f} = 0$. by $e^{\gamma x}$,

$$\Rightarrow \quad \xi_1 \frac{\partial}{\partial x} e^{\gamma x} \tilde{f} - \gamma \xi_1 e^{\gamma x} \tilde{f} + \mathcal{L} e^{\gamma x} \tilde{f} = 0.$$

Energy estimate. $\Rightarrow \frac{1}{2} \frac{d}{dx} e^{2\gamma x} \int \xi_1 w^2 d\xi + e^{2\gamma x} \int (-\gamma \xi_1 + \nu_0 (1+|\xi|)) w^2 d\xi \le 0.$

Since, $e^{2\gamma x} \int \xi_1 w^2 d\xi \ge 0$

$$\Rightarrow. \quad -\frac{1}{2} \int \xi_1 w^2 d\xi (x=0) + \int_0^\infty \int e^{2\gamma x} (-\gamma \xi_1 + \nu_0 (1+|\xi|)) w^2 d\xi dx \le 0.$$

If $\gamma < \nu_0$,
$$\iint e^{2\gamma x} (1+|\xi|) w^2 d\xi dx \le \frac{1}{2} \int \xi_1 w^2 d\xi (x=0).$$

$$< (\nu_0 - \gamma)^{-1} [K_g + m_f^2].$$

Pointwise decay of w:

Assumption: $\tilde{f}_x \in L^2_{loc}(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3)) \Rightarrow K\tilde{f}_x \in L^2_{loc}(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3))$:

Thus $(\xi_1 \frac{d}{dx} + \nu)\tilde{f}_X \in L^2_{loc}(\mathbb{R}^+_x, L^2(\mathbb{R}^3_\xi))$.

$$\xi_1 \frac{d}{dx} \tilde{f}_X + L \tilde{f}_X = 0$$

be smooth cutoff function ϕ with $\phi(x) = 0$; $0 \le x \le \frac{1}{2}Y$ or $X+1 < x < \infty$

$\phi(x) = 1$ for $Y \le x < X$.

$(\phi f)_X$ satisfies. $\xi_1 \frac{d}{dx}(e^{\gamma x}\phi \tilde{f})_X + L(e^{\gamma x}\phi w)_X = \xi_1 (e^{\gamma x}\phi)_{xx}\tilde{f} - (e^{\gamma x}\phi)_x L w$.

Energy estimate $\Rightarrow$.

$$\frac{1}{2}\frac{d}{dx}\int \xi_1 ((e^{\gamma x}\phi \tilde{f})_X)^2 d\xi + (\nu_0 - \gamma)\int (1+|\xi|)[(e^{\gamma x}\phi w)_X]^2 d\xi$$

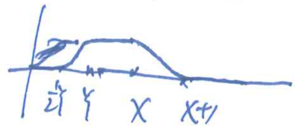
$$\le \int (e^{\gamma x}\phi \tilde{f})_X \cdot \{\xi_1 (e^{\gamma x}\phi)_{xx}\tilde{f} - (e^{\gamma x}\phi)_x L w\} d\xi.$$

$\tilde{q}\,\tilde{q}_X$ vanish

$$= \int (e^{\gamma x}\phi w)_X \{\xi_1 (e^{\gamma x}\phi)_{xx} w - (e^{\gamma x}\phi)_x L w\} d\xi.$$

$$\le k\left(\int (1+|\xi|)(e^{\gamma x}\phi w)_X^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int (1+|\xi|)e^{2\gamma x} w^2 d\xi\right)^{\frac{1}{2}}.$$

↳ independent of X. $\phi(x)$:



Integration over x:

$$(\nu_0 - \gamma)\int_0^\infty \int (1+|\xi|)(e^{\gamma x}\phi w)_X^2 d\xi dx.$$

$$\le k\left(\int_0^\infty \int (1+|\xi|)(e^{\gamma x}\phi w)_X^2 d\xi dx\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty \int (1+|\xi|)e^{2\gamma x} w^2 d\xi dx\right)^{\frac{1}{2}}.$$

$$\Rightarrow \int_0^\infty \int c(1+|\xi|)(e^{\gamma x}\phi w)_x^2 \, d\xi dx \le K(\nu_0-\gamma)^{-3}.$$

~~$\phi=1$ for~~ K is independent of X, $\phi=1$, take $X\to\infty$.

$$\int_Y^\infty \int c(1+|\xi|)e^{2\gamma x} w_x^2 \, d\xi dx \le K(\nu_0-\gamma)^{-3}.$$

$$e^{2\gamma X}\int c(1+|\xi|)w(X)^2 d\xi = 2\int_0^X \int c(1+|\xi|)(e^{\gamma x}\phi w)(e^{\gamma x}\phi w)_x \, d\xi dx.$$

$$\lesssim \int_0^\infty \int c(1+|\xi|)(e^{\gamma x}\phi w)_x^2 \, d\xi dx.$$

$$\int_0^\infty \int e^{2\gamma x} c(1+|\xi|)w^2 d\xi dx \le K(\nu_0-\gamma)^{-2}. \quad (*)$$

Asymptotic behavior of $\tilde{q}$, $\tilde{q} = (a+b_2\xi_2+b_3\xi_3+c\xi^2)M^{\frac12}$.

Multiply $\left[\xi_1 \frac{\partial}{\partial x}\tilde{f} + \mathcal{L}\tilde{f} = 0\right]$. $\xi_1(1,\xi_2,\xi_3,\xi^2)M^{\frac12} = \Psi$

$\Rightarrow$ $A=(a,b_2,b_3,c)^T$, $\Updownarrow$ $\xi_1\frac{\partial}{\partial x}\tilde{q} + \mathcal{L}w = -\xi_1\frac{\partial}{\partial x}w$

$$\Rightarrow \quad S_1\frac{d}{dx}A + \int \Psi \mathcal{L}w \, d\xi = \frac{-d}{dx}\int \xi_1\Psi_1 w \, d\xi.$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 35 \end{pmatrix}$$

$$\left|\int \Psi_1 \mathcal{L}w \, d\xi\right|, \quad \left|\int \xi_1\Psi_1 w \, d\xi\right| \le K(\nu_0-\gamma)^{-1}e^{-\gamma x} \quad \text{by } (*),$$

S_1 positive definite, a, b_2, b_3, c approaches limit $(a_\infty, b_{2\infty}, b_{3\infty}, c_\infty)$ at rate $e^{-\gamma x}$.

To compute $(a_{\infty}, b_{2\infty}, b_{3\infty}, c_{\infty})$,

multiply $f(x=0)$ by $\Psi_2 = (1, \xi_2, \xi_3, \xi^2)^T M^{1/2}$ and $\int$ over $\xi_1 > 0$,

$$S_2 A(x=0) = \int_{\xi_1>0} (g-w)\Psi_2\, d\xi \le k.$$

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 15 \end{pmatrix} \Rightarrow |a_{\infty}| + |b_{2\infty}| + |b_{3\infty}| + |c_{\infty}| < k.$$

Existence: $\mathbb{R}^3_N = \{\xi : |\xi_1| > \frac{1}{N},\ |\xi| < N\}$.

$$\chi_N = \begin{cases} 1 & \text{if } |\xi_1| > N^{-1}\ \&\ |\xi| < N \\ 0, & \text{otherwise}. \end{cases}$$

$\psi_{iN} = \chi_N \psi_i$, $\quad P_N =$ Projection onto $\{\psi_{iN}, i=0,\dots,4\}$.

Denote $L_N = \chi_N (I-P_N) L (I-P_N) \chi_N = \nu \chi_N + K_N$

$$K_N = \chi_N(-\nu P_N - P_N \nu + P_N \nu P_N)\chi_N + \chi_N (I-P_N) K (I-P_N)\chi_N$$

L_N is still non-negative, self-adjoint, non bdd operator on L^2.

null space $N(L_N)$ spanned by ψ_{iN}, $i=0,\dots,4$. K_N is compact on L^2.

Proposition: $\exists$ a soln $f = f^{(N)} \in L^2([0,1] \times \mathbb{R}^3_N)$ to

$$\begin{cases} \xi_1 \frac{\partial f}{\partial x} + L_N f = 0, & 0 \le x \le 1 \\ f = g, & x=0,\ \xi_1 > 0. \\ f(\xi_1,\xi_2,\xi_3) = f(-\xi_1,\xi_2,\xi_3) & x = 1. \\ mf = 0 & \end{cases}$$

$f^{1r} = w + q, \quad q = \chi_r(\xi)\,(a + b_2\xi_2 + b_3\xi_3 + c\xi^2)\,M^{1/2}(\xi).$

$\exists\ a_0, b_{20}, b_{30}, c_0 < k$ s.t.

$$\int_{\mathbb{R}^3_r} (1+|\xi|)\,w^2\,d\xi + (a-a_0)^2 + (b_2-b_{20})^2 + (b_3-b_{30})^2 + (c-c_0)^2 \le k\,(r_0-r)^{-2}\,e^{-2rx}.$$

Transport equation with diffuse boundary condition..

$$\partial_t F + v\cdot\nabla_x F = 0 \quad , (t,x,v)\in\mathbb{R}_+\times\Omega\times\mathbb{R}^3 .$$

Ω: smooth and strictly convex.

$$F = \mu\int_{n\cdot v_1>0} F(t,x,v_1)\,(n(x)\cdot v_1)\,dv_1$$

Perturbation: $f(t,x,v) = F(t,x,v) - \mu(v)$.

Goal: under certain assumption on the initial data,

$\exists!\ f$ s.t. $\sup_{t\geq 0}\|e^{\theta'|v|^2} f(t)\|_{L^\infty_{x,v}} \leq C\|e^{\theta'|v|^2} f_0\|_{L^\infty_{x,v}}$.

and $\sup_{x\in\Omega}\int_{\mathbb{R}^3} e^{\theta|v|^2}|f(t,x,v)|\,dv \leq C_\theta \langle t\rangle^{-3}(\ln\langle t\rangle)^2 \quad t\geq 0$.

Decay of the L^1 norm

Lemma: ~~suppose~~ $\exists\ m(x,v)\geq 0$ such that for $\forall\ T_0>>1$, N,

$$f(NT_0,x,v) \geq m(x,v)\Big\{\iint_{\Omega\times\mathbb{R}^3} f((N-1)T_0,x,v)\,dx\,dv$$

$$-\iint_{\Omega\times\mathbb{R}^3} 1_{t_f(x,v)\geq\frac{3T_0}{4}} f((N-1)T_0,x,v)\,dv\,dx\Big\}.$$

Proof: iterate along characteristic

$$\Rightarrow \quad f(NT_0, x, v) \geq 1_{t_b(x,v) \leq \frac{T_0}{4}} \mu(v) \int_{V_1}\int_{V_2}\int_{V_3} 1_{t_3 \geq (N-1)T_0} \cdot f(t_3, x_3, v_3)\{n(x_3)\cdot v_3\}\, dv_3\, d\sigma_2\, d\sigma_1$$

change of variable

$$\geq 1_{t_b(x,v) \leq \frac{T_0}{4}} \mu(v) \int_0^{t - t_b(x,v)} \int_{\partial\Omega} \frac{|n(x_2)\cdot(x_1 - x_2)|}{|t_{b,1}|^4} \frac{|n(x_1)\cdot(x_1 - x_2)|}{t_{b,1}} \mu\left(\frac{|x_1 - x_2|}{t_{b,1}}\right)$$

$$\times \int_0^{t - t_b(x,v) - t_{b,1}} \int_{\partial\Omega} \frac{|n(x_3)\cdot(x_2 - x_3)|}{|t_{b,2}|^4} \frac{|n(x_2)\cdot(x_2 - x_3)|}{t_{b,2}} \mu\left(\frac{|x_2 - x_3|}{t_{b,2}}\right)\cdot$$

$$1_{t_3 \geq (N-1)T_0} \int_{n(x_3)\cdot v_3 > 0} f(t_3, x_3, v_3)(n(x_3)\cdot v_3)\, dv_3\, dS_{x_3}\, dt_{3,2}\, dS_{x_2}\, dt_{3,1}.$$

Restrict range of x_2 as; $\delta > 0$

$$X_2^\delta = \{x_2 \in \partial\Omega : |x_1 - x_2| > \delta \text{ and } |x_2 - x_3| > \delta\}.$$

define $t_+ = t_{b,1} + t_{b,2} \in [0, T_0 - t_b(x,v)]$, $t_- = t_{b,1} - t_{b,2}$
$\in [-(T_0 - t_b(x,v)), T_0 - t_b]$

Introduce $A_+ = \{t_+ : T_0 - t_b - \min(t_b(x_3,v_3), \frac{T_0}{4}) \leq t_+ \leq T_0 - t_b(x,v)\}$.

$A_- := \{t_- : |t_-| \leq T_0 - t_b(x,v) - \min(t_b(x_3,v_3), \frac{3T_0}{4})$