Solution 9

1. It suffices to show that

$$\langle \nabla_g i\left(x^*\right), x - x^* \rangle \leq 0$$

for all i such that $g_i\left(x^*\right) = 0$. Fix such an i . Define $h(t) = g_i\left((1-t)x^* + tx\right)$. Then $h(0) = 0$ and $h'(0) = \langle \nabla g_i\left(x^*\right), x - x^* \rangle$. Since $g_i(y) \leq 0$ for all feasible y, we have $h(t) \leq 0$ for all $t \in [0,1]$, so $h'(0) \leq 0$.

2. (a) The feasible region is a nonempty compact set and the objective function is continuous, so an optimal solution exists.

(b)

$$\begin{aligned} -[2,3,2] - \lambda[1,1,1] + \mu[2x,2y,2z] &= [0,0,0] \\ x + y + z &\geq 0 \\ x^2 + y^2 + z^2 &= 1 \\ \lambda &\geq 0 \\ \lambda(x+y+z) &= 0 \end{aligned}$$

(c) By KKT, $\mu \neq 0$ and

$$[x,y,z] = \frac{1}{2\mu}[\lambda+2, \lambda+3, \lambda+2]$$

Again by KKT, we have $\mu > 0$. If $\lambda = 0$,

$$\begin{aligned} \mu &= \sqrt{1^2 + \frac{3^2}{2^2} + 1^2} \\ [x,y,z] &= \frac{1}{\sqrt{17}}[2,3,2] \\ 2x + 3y + 2z &= \sqrt{17} \end{aligned}$$

If $\lambda \neq 0$, then $x + y + z = 0$, so

$$\begin{aligned} 3\lambda + 7 &= 2\mu \\ 3\lambda^2 + 14\lambda + 17 &= 4\mu^2 \end{aligned}$$

which implies $\lambda = -2$ or $\lambda = -\frac{5}{3}$, a contradiction. Thus,

$$(\lambda^*, \mu^*, x^*, y^*, z^*) = \left(0, \frac{\sqrt{17}}{2}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}\right).$$

3. (a)

$$\begin{aligned} 2Ax + 2\mu x &= 0 \\ \|x\|^2 &= 1 \end{aligned}$$

(b) Suppose x minimizes $\langle x, Ax\rangle$ on $\|x\|^2 = 1$. Then x necessarily satisfies KKT. In particular, x is a nonzero vector such that $(A + \mu I)x = 0$ for some $\mu \in \mathbb{R}$, so x is an eigenvector of A with eigenvalue $-\mu$.