Continuity

Def: Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ & $\vec{a} \in A$ (so $f(\vec{a})$ is defined)

Then f is said to be <u>continuous at $\vec{a}$</u>

if $\boxed{\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})}$ (exists & equal to $f(\vec{a})$)

Equivalently, $\forall \varepsilon > 0, \exists \delta > 0$ such that

if $\vec{x} \in A$ & $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

Def $f: A \to \mathbb{R}$ is said to be <u>continuous</u> (on A) if

f is <u>continuous at every point</u> in A.

eg: Let $k = 1, \cdots, n$. Show that $f(x_1, \cdots, x_n) = x_k$ is continuous on $\mathbb{R}^n$

(usually called the k-th coordinate function)

Pf: Let $\vec{a} = (a_1, \cdots, a_k, \cdots, a_n) \in \mathbb{R}^n$

$\forall \varepsilon > 0$, take $\delta = \varepsilon$

Then (for $\vec{x} \in \mathbb{R}^n$) $\|\vec{x} - \vec{a}\| < \delta$ implies

$$|f(\vec{x}) - f(\vec{a})| = |x_k - a_k| \le \sqrt{(x_1 - a_1)^2 + \cdots + (x_k - a_k)^2 + \cdots + (x_n - a_n)^2}$$

$$< \delta = \varepsilon$$

Hence $f(\vec{x}) = x_k$ is continuous at $\vec{a}$.

Since $\vec{a} \in \mathbb{R}^n$ is arbitrary, we've proved that f is cts. on $\mathbb{R}^n$ #

Thm If $f, g : A \overset{\subset \mathbb{R}^n}{\longrightarrow} \mathbb{R}$ are <u>continuous</u> at $\vec{a} \in A$, then

(1) $f(\vec{x}) \pm g(\vec{x})$, $kf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are <u>continuous</u> at $\vec{a}$,
where k is a constant.

(2) $\dfrac{f(\vec{x})}{g(\vec{x})}$ is <u>continuous</u> at $\vec{a}$ provided $g(\vec{a}) \neq 0$

<u>Consequences</u>:

(i) All polynomials of multi-variables are continuous (on $\mathbb{R}^n$)

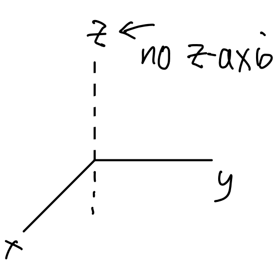
(ii) All rational functions of multi-variables are continuous on their domain of definition

(rational function $\overset{\text{def}}{=} \dfrac{P(\vec{x})}{Q(\vec{x})}$ for some polynomials $P(\vec{x})$ & $Q(\vec{x})$
"domain of definition" of $\dfrac{P(\vec{x})}{Q(\vec{x})} = \mathbb{R}^n \setminus \{\vec{x} : Q(\vec{x}) = 0\}$)

<u>egs</u> (1) $x^3 + 3yz + z^2 - x + 7y$ is a polynomial on $\mathbb{R}^3$
& is continuous on $\mathbb{R}^3$

(2) $\dfrac{x^3 + y^2 + yz}{x^2 + y^2}$ is a rational function on $\mathbb{R}^3$

domain of definition $= \mathbb{R}^2 \setminus \{(0,0,z)\}$
$= \mathbb{R}^3 \setminus \{z\text{-axis}\}$



& is continuous on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$.

Fact: Let $\vec{a}$ be a zero of polynomial $Q(\vec{x})$ (ie. $Q(\vec{a})=0$)
Then the rational function $r(\vec{x}) = \frac{P(\vec{x})}{Q(\vec{x})}$ can be

"extended to a function continuous at $\vec{a}$ $\iff$ $\lim_{\vec{x}\to\vec{a}} r(\vec{x})$ exists".

egs (1) $f(x,y) = \frac{xy+y^3}{x^2+y^2}$ (in $\mathbb{R}^2$)

Note $x^2+y^2=0 \iff (x,y)=(0,0)$
$\therefore$ Domain of definition of f is $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy+y^3}{x^2+y^2} = \lim_{x\to 0} \frac{x(mx)+(mx)^3}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{m+m^3x}{1+m^2} = \frac{m}{1+m^2}$$

different limits in different directions

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ (x,y)\neq(0,0)}} \frac{xy+y^3}{x^2+y^2} \text{ DNE}$$

$\therefore$ $\frac{xy+y^3}{x^2+y^2}$ cannot be extended to a function cts. at $(0,0)$.

$\#$

(2) $g(x,y) = \frac{x^4-y^4}{x^2+y^2}$ (Note $x^2+y^2=0 \iff (x,y)=(0,0)$)

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{r\to 0} \frac{r^4(\cos^4\theta - \sin^4\theta)}{r^2} = \lim_{r\to 0} r^2(\cos^4\theta-\sin^4\theta)$$

$$= 0 \quad \text{(by Squeeze Thm)}$$

$\therefore \quad g(x,y) = \dfrac{x^4-y^4}{x^2+y^2}$ can be extended to a function continuous at $(0,0)$.

In fact $g(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases} \quad \left(= x^2-y^2\right)$

is the required extension.

Thm If • $f: A \xrightarrow{} \mathbb{R}$, $A \subset \mathbb{R}^n$, is <u>continuous at $\vec{a} \in A$</u>,

• $g(x)$ is a 1-variable function <u>continuous at $f(\vec{a})$</u>

Then $g\circ f(\vec{x}) \left(\overset{\text{def}}{=} g(f(\vec{x}))\right)$ is continuous at $\vec{a}$, and

$$\lim_{\vec{x}\to\vec{a}} g(f(\vec{x})) = g\left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right) = g(f(\vec{a})).$$

<u>egs</u>: (1) x_k k-th coordinate function are continuous, $\forall k=1,\cdots,n$

• $|x|$ is also continuous $\Rightarrow$ $|x_k|$ are continuous.

• $\ln|x|$ is continuous for $|x|>0 \Rightarrow \ln|x_k|$ are continuous if $|x_k|>0$.

(2) $\sin(x^2+yz)$, e^{x-y}, $\cos\left(\frac{1}{x^2+y^2}\right)$ (except $(x,y)=(0,0)$)

$r=\sqrt{x^2+y^2}$ are continuous on their domains.

Partial Derivatives

Def: Let
- $\Omega \subseteq \mathbb{R}^n$ be open
- $f : \Omega \to \mathbb{R}$ be a function

Then the <u>i-th partial derivative</u> of f at $\vec{x} = (x_1, \cdots, x_n) \in \Omega$ is defined by

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n)$$

$$= \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)}{h}$$

(provided the limit exists)

Remarks: (1) Ω open "ensures" $(x_1, \cdots, x_i + h, \cdots, x_n) \in \Omega$ for small h so that $f(x_1, \cdots, x_i + h, \cdots, x_n)$ is defined.

(2) If $n = 1$, $\frac{\partial f}{\partial x} = \frac{df}{dx}$

(3) If $n = 2$, we usually write

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

(4) In practice, $\frac{\partial f}{\partial x_i}$ can be calculated as derivative in one variable x_i by regarding other variables as constants.

(5) Other notations:
$$\frac{\partial f}{\partial x} = \partial_1 f = D_1 f = \nabla_1 f = f_x$$
$$\frac{\partial f}{\partial y} = \partial_2 f = D_2 f = \nabla_2 f = f_y$$

eg $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x + 0 = 2x$$

regarded as constant $\qquad \frac{\partial y^2}{\partial x} = 0 \qquad \frac{\partial x^2}{\partial y} = 0$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 0 + 2y = 2y$$

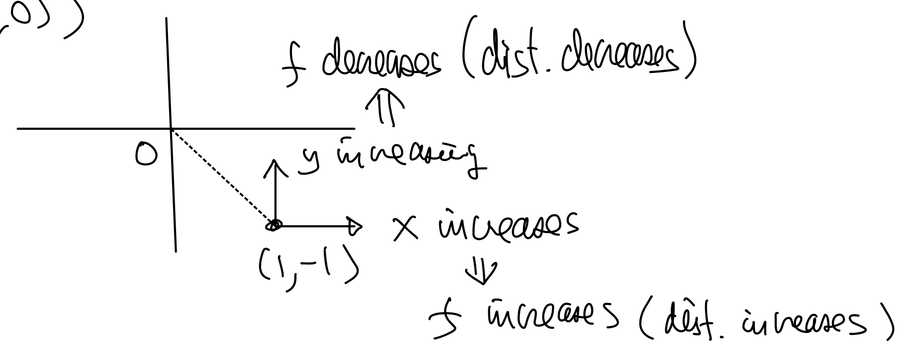
Note: As the point $(1,-1)$

$$\frac{\partial f}{\partial x}(1,-1) = 2 > 0 \quad \& \quad \frac{\partial f}{\partial y}(1,-1) = -2 < 0$$

f increases as x increases at $(1,-1)$

f decreases as y increases at $(1,-1)$

$f(x,y) = x^2+y^2 = (\text{dist. to } (0,0))^2$



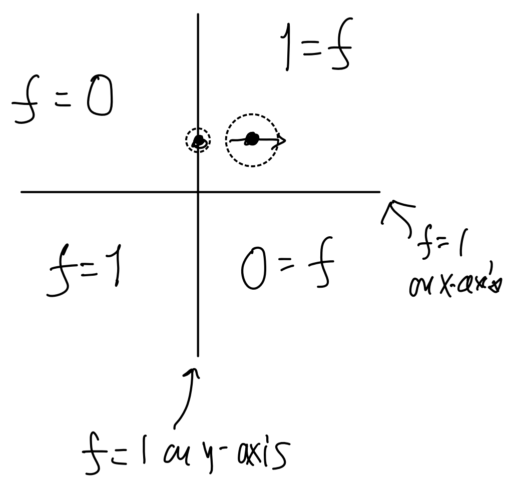
eg: $f(x,y,z) = xy^2 - \cos(xz)$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xy^2 - \cos(xz)\right) = y^2 + z\sin(xz)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy^2 - \cos(xz)\right) = 2xy$$

$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xy^2 - \cos(xz)\right) = x\sin(xz)$$ ※

eg $f(x,y) = \begin{cases} 1 & , \text{if } xy \geq 0 \\ 0 & , \text{if } xy < 0 \end{cases}$



Find $\frac{\partial f}{\partial x}(1,1)$, $\frac{\partial f}{\partial x}(0,1)$, & $\frac{\partial f}{\partial x}(0,0)$

Soln: For $(1,1)$

(|h| small) $$\lim_{h\to 0} \frac{f(1+h,1) - f(1,1)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\therefore \frac{\partial f}{\partial x}(1,1) = 0$$

For $(0,0)$

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\therefore \frac{\partial f}{\partial x}(0,0) = 0$$

For $(0,1)$

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(0+h,1) - f(0,1)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{1-1}{h} = 0$$

$$\lim_{\substack{h\to 0\\ h<0}} \frac{f(0+h,1)-f(0,1)}{h} = \lim_{\substack{h\to 0\\ h<0}} \frac{0-1}{h} \quad \text{DNE}$$

$$\therefore \quad \frac{\partial f}{\partial x}(0,1) \text{ DNE}.$$

(<u>Note</u>: $\frac{\partial f}{\partial x}(0,0)$ exists, but f is <u>not continuous</u> at $(0,0)$

<u>check</u>: $\frac{\partial f}{\partial y}(0,0) = 0$ (Optional Ex.))