THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 10
Due Date: 25th April 2024

Compulsory Part

1. Find an irreducible polynomial $p \in \mathbb{Q}[x]$ such that:
   (a) $\mathbb{Q}[x]/(p) \cong \mathbb{Q}(2-\sqrt{2})$.
   (b) $\mathbb{Q}[x]/(p) \cong \mathbb{Q}\left(\sqrt{1+\sqrt{3}}\right)$.
   (c) $\mathbb{Q}[x]/(p) \cong \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

2. (a) Show that x^2-5 is irreducible in $\mathbb{Q}(\sqrt{2})[x]$.
   (b) Show that $\mathbb{Q}(5+\sqrt{2}) = \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(2+\sqrt{5}) = \mathbb{Q}(\sqrt{5})$ as subfields of $\mathbb{R}$.
   (c) Show that $\sqrt{5}$ does not lie in $\mathbb{Q}(\sqrt{2})$.
   (d) Conclude that $5+\sqrt{2}$ and $2+\sqrt{5}$ cannot be roots of the same irreducible polynomial in $\mathbb{Q}[x]$.

3. Consider the subfield $F = \mathbb{Q}(\sqrt[3]{5})$ of $\mathbb{R}$. Express the multiplicative inverse of $2+\sqrt[3]{5} \in F$ in the form:
$$a + b\gamma + c\gamma^2,$$
where $a, b, c \in \mathbb{Q}$ and $\gamma = \sqrt[3]{5}$.

4. Find an irreducible polynomial of degree 3 in $\mathbb{F}_2[x]$, and hence construct a finite field with 8 elements.

Optional Part

1. Let F be a subfield of a field E, and γ an element in E. Show that $F(a+b\gamma) = F(\gamma)$ for all nonzero $a, b \in F$.

2. Let F be a subfield of a field E, and γ an element in E. Let p, q be irreducible polynomials in $F[x]$ such that γ is a root of both p and q. Show that $q = up$ for some nonzero element $u \in F$.

3. Let $p = x^3 - x^2 + 1 \in \mathbb{F}_3[x]$.
   (a) Show that $K := \mathbb{F}_3[x]/(p)$ is a field.
   (b) Express the multiplicative inverse of $x^2+1+(p) \in K$ in the form:
$$a + bx + cx^2 + (p),$$
   where $a, b, c \in \mathbb{F}_3$.