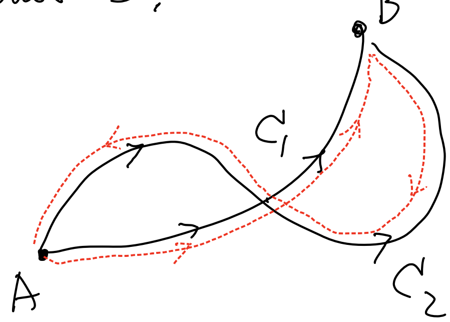
(Cont'd)

"(b) ⇒ (c)" Suppose C_1, C_2 are oriented curves with starting point A and end point B.



Then $C_1 \cup (-C_2)$

$= C_1 - C_2$ (a better notation)

is an oriented closed curve.

Then by (b)

$$0 = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}$$

Since C_1 & C_2 are arbitrary, $\vec{F}$ is conservative.

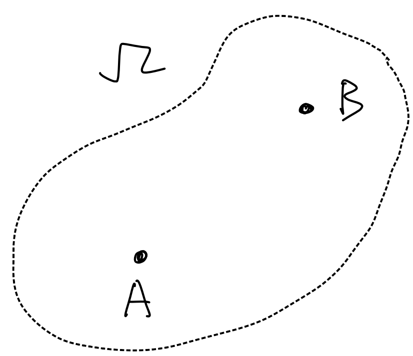
"(c) ⇒ (a)"

Assume $n=2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ be conservative.

Fix a point $A \in \Omega$

Then for any point $B \in \Omega$, define



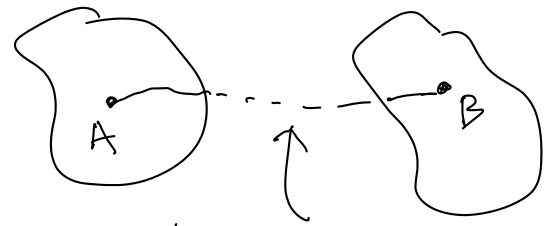
$$f(B) = \int_A^B \vec{F} \cdot \hat{T} \, ds = \underline{\text{common value}} \text{ of } \int_C \vec{F} \cdot \hat{T} \, ds$$

($\vec{F}$ is conservative)

for any C from A to B.

Since $\vec{F}$ is conservative, $f(B)$ is well-defined.

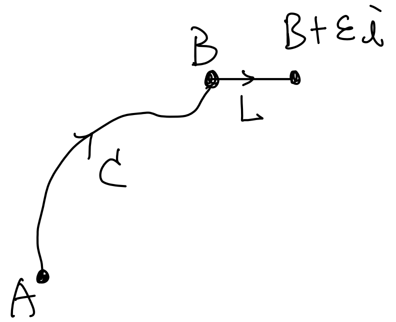
[We've also used the assumption that Ω is <u>connected</u>, otherwise there is no path from A to B, if A, B belong to different connected components:



]

<u>Claim</u> $\vec{F} = \vec{\nabla} f$

<u>Pf of claim</u>: $\displaystyle \frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon}$



horizontal line segment from B to $B+\varepsilon\hat{i}$ with $|\varepsilon|$ sufficiently small s.t. $C + L \subset \Omega$ (open)

Let C be an oriented curve from A to B. Then

$$f(B+\varepsilon\hat{i}) = \int_A^{B+\varepsilon\hat{i}} \vec{F} \cdot d\vec{r}$$

$$= \int_{C+L} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r}$$

$$= \int_A^B \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r}$$

$$= f(B) + \int_L \vec{F}\cdot d\vec{r}$$

$$\therefore \quad \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon} = \frac{1}{\varepsilon}\int_L \vec{F}\cdot d\vec{r}$$

$$= \frac{1}{\varepsilon}\int_0^{\varepsilon} M(x+t, y)\,dt$$

(parametize L by $B + t\hat{i}$, $t \in [0, \varepsilon]$, $B = (x, y)$)

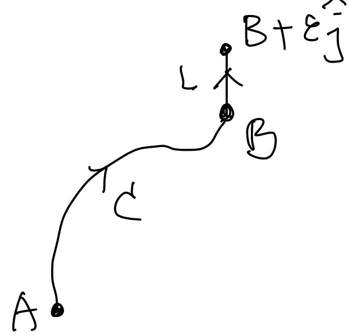
$$\Rightarrow \quad \frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^{\varepsilon} M(x+t, y)\,dt$$

($\vec{F}$ is continuous)

$= M(x,y)$ (by MVF & M is continuous)
(or Fundamental Thm of Calculus)

Similarly $\frac{\partial f}{\partial y}(B) = N(x,y)$

by consider:



(vertical line segment)

So $\vec{\nabla} f = \vec{F}$

Since $\vec{F}$ is continuous, $M = \frac{\partial f}{\partial x}$ & $N = \frac{\partial f}{\partial y}$ are continuous

$\Rightarrow f \in C^1$ ##

Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

connected open

"$n=3$" If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ (on $\Omega \subset \mathbb{R}^3$)

then
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y} \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$

connected open

"$n=2$" If $\vec{F} = M\hat{i} + N\hat{j}$ (on $\Omega \subset \mathbb{R}^2$)

then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Pf: $\vec{F}$ conservative $\xrightarrow{\text{Thm 9}}$ $\vec{F} = \vec{\nabla} f$ for some function f.

i.e. $\vec{\nabla} f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j} + \dfrac{\partial f}{\partial z}\hat{k}$

$= M\hat{i} + N\hat{j} + L\hat{k}$

$\vec{F} \in C^1 \Rightarrow f \in C^2$

Hence mixed derivatives thm (Clairaut's Thm)

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial L}{\partial y} \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial M}{\partial z} \end{cases}$$

(included "$n=2$" case)

#

eg 42: Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is not conservative in $\mathbb{R}^2$.

Soln: ($\vec{F} \in C^\infty$)
$$\begin{cases} M = 1 \\ N = x \end{cases} \Rightarrow \begin{cases} \dfrac{\partial M}{\partial y} = 0 \\ \quad\quad \neq \\ \dfrac{\partial N}{\partial x} = 1 \end{cases}$$

By Cor to Thm 9, $\vec{F}$ is not conservative.

Remark (Important)

For a C^1 vector field $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$\boxed{\vec{F} \text{ conservative}} \quad \underset{?}{\overset{\text{Cor. to Thm 9}}{\rightleftarrows}} \quad \boxed{\begin{array}{c} M, N, L \text{ satisfy the system} \\ \text{of PDEs in the Cor to Thm 9} \end{array}}$$

Answer: NOT TRUE in general, needs extra condition on the domain Ω ("connected" is not enough)

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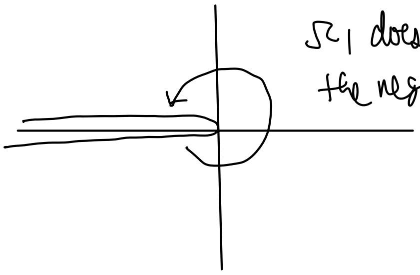
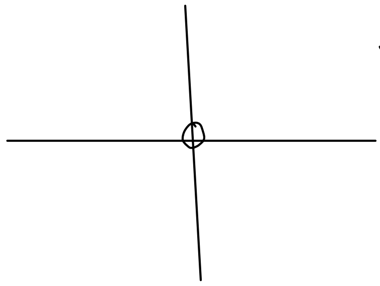
eg 43 Consider the vector field
$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

and the domains $\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$
$$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$$



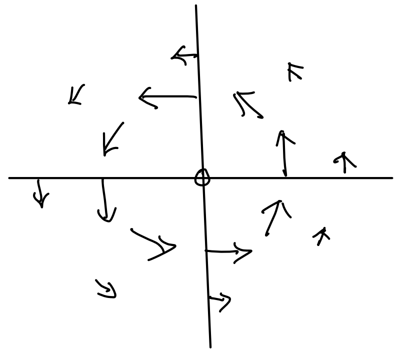
Ω_1 doesn't include the negative X-axis (and origin)



Ω_2 doesn't include the origin (but include the negative X-axis.)

In polar coordinates

$$\vec{F} = -\frac{\sin\theta}{r}\,\hat{i} + \frac{\cos\theta}{r}\,\hat{j}$$



$$\Rightarrow \begin{cases} \cdot\ \vec{F} \text{ rotates around the origin anti-clockwisely} \\ \cdot\ |\vec{F}| = \frac{1}{r} \longrightarrow 0 \quad \text{as } r \to +\infty \\ \cdot\ |\vec{F}| = \frac{1}{r} \longrightarrow +\infty \quad \text{as } r \to 0 \end{cases}$$

$\Rightarrow$ $\vec{F}$ cannot be extended to a C^1 vector field on $\mathbb{R}^2$.

Besides $(0,0)$, $\vec{F}$ is C^1, hence

$\vec{F}$ is C^1 on Ω_1, and also

$\vec{F}$ is C^1 on Ω_2.

Questions : Is $\vec{F}$ conservative on Ω_1?

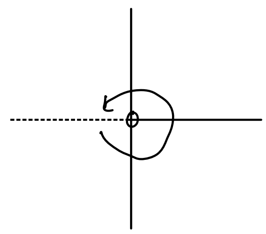
Is $\vec{F}$ conservative on Ω_2?

Soln: (1) For Ω_1, and $(x,y) \in \Omega_1$ can be expressed in polar coordinates by

$$\begin{cases} r > 0 \\ -\pi < \theta < \pi \end{cases} \qquad (r,\theta) \text{ are unique}$$

doesn't include end values.

Define $f(x,y) = \theta$ "smooth" on Ω_1



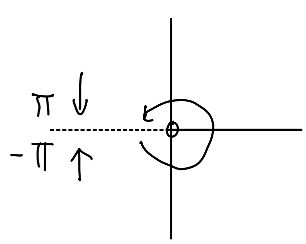
Then $$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r} \end{cases}$$ (check!)

$$\Rightarrow \quad \vec{F} = -\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= \vec{\nabla} f \quad \text{on } \Omega_1$$

$\Rightarrow$ $\vec{F}$ conservative on Ω_1

(2) For Ω_2, the "function" $f(x,y) = \theta$ <u>cannot be extended</u> to a "smooth" function on (the whole) Ω_2.



the "function" $f = \theta$ "jumps" at the negative x-axis

$\Rightarrow$ f cannot be extended to a continuous function across negative x-axis.

To show that $\vec{F}$ is not conservative on Ω_2,

we consider a closed curve

$C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \qquad t \in [-\pi, \pi]$

(unit circle in Ω_2, but it is not a curve in Ω_1)

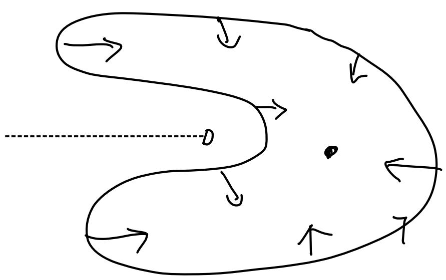
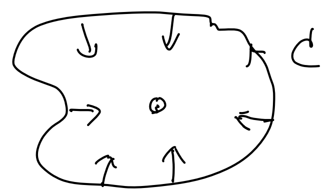
Then $\oint_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} \left(-\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}\right) \cdot \vec{r}\,'(t)\,dt$ ($r = 1$ on C)

$$= \int_{-\pi}^{\pi} (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \cdot (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,dt$$

$$= \int_{-\pi}^{\pi} 1\,dt = 2\pi \neq 0$$

By Thm 9, $\vec{F}$ is not conservative on Ω_2 #

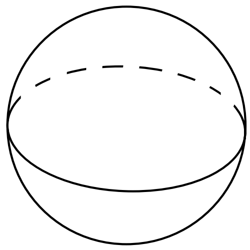
<u>Summary</u>

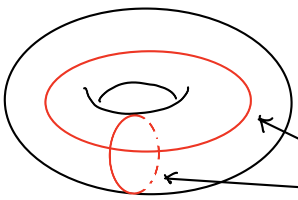
Ω_1	Ω_2
$f(x,y)=\theta$ smooth function on Ω_1	$f(x,y)=\theta$ is <u>not</u> a smooth function on Ω_2 (θ cannot be well-defined on the whole Ω_2)
$C: x^2+y^2=1$ is <u>not</u> a curve in Ω_1 ($(-1,0)\in C$ but $(-1,0)\notin\Omega_1$)	$C: x^2+y^2=1$ is a closed curve in Ω_2
 closed curve <u>cannot</u> circle around the origin $\Rightarrow$ closed curves can be deformed continuously (within Ω_1) to a point (in Ω_1)	 C encloses the "hole" $\Rightarrow$ C cannot be deformed continuously (within Ω_2) to a point (in Ω_2)

<u>Def 15</u> A subset $\Omega\subset\mathbb{R}^n$, $n=2$ or 3, is called <u>simply-connected</u> if every closed curve in Ω can be <u>contracted</u> to a point in Ω without ever leaving Ω.

(contracted := deformed continuously)

eg44 Ω_1 in eg43 is simply-connected, but Ω_2 is not simply-connected.

eg45:  $S^2 \subset \mathbb{R}^3$ $S^2 = \{x^2+y^2+z^2=1\}$ (unit sphere) is simply-connected.

eg46: 

torus $\mathbb{T}^2 \cong S^1 \times S^1 \subset \mathbb{R}^3$ is not simply-connected

these 2 closed curves cannot be contracted to a point on $\mathbb{T}^2$.

Remark: Simply-connectedness is a global condition to guarantee " "PDEs in Cor to Thm 9" $\Rightarrow$ "conservative" "

Thm 10: Suppose $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, is connected and simply-connected. Let $\vec{F}$ be C^1 vector field on Ω.

Then

$\vec{F}$ is conservative on Ω $\Leftrightarrow$ components of $\vec{F}$ satisfy the system of PDEs in the Cor to Thm 9.

(Pf: later)

eg 47 : Let $\Omega \equiv \mathbb{R}^3$ (connected and simply-connected)

Let $\vec{F} = M\vec{i} + N\vec{j} + L\vec{k}$

$$= (y+e^z)\vec{i} + (x+1)\vec{j} + (1+xe^z)\vec{k}.$$

Find the potential function f of $\vec{F}$, i.e. $\vec{\nabla} f = \vec{F}$

Soln This is, we want to solve

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L.$$

Checking M, N, L satisfy the system of PDEs in the Cor to Thm 9 :

$$\frac{\partial M}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial M}{\partial z} = e^z$$

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial N}{\partial y} = 0 \qquad \frac{\partial N}{\partial z} = 0$$

$$\frac{\partial L}{\partial x} = e^z \qquad \frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial z} = xe^z$$

(In fact, no need to find $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}, \frac{\partial L}{\partial z}$)

Thm 10 $\Rightarrow$ existence of potential function f.

To find f explicitly

$$\frac{\partial f}{\partial x} = M = y + e^z$$

$$\Rightarrow \quad f = \int (y+e^z)\,dx$$

$$= x(y+e^z) + \text{"const in } x\text{"}$$

(could be a function of y, z)

$$\Rightarrow \quad f = x(y+e^z) + g(y,z) \quad \text{for some function } g(y,z)$$

Then take $\frac{\partial}{\partial y}$,

$$N = x+1 = \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z)$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = y + h(z) \quad \text{for some function } h(z)$$

$$\Rightarrow \quad f = x(y+e^z) + y + h(z)$$

Then take $\frac{\partial}{\partial z}$,

$$L = 1 + xe^z = \frac{\partial f}{\partial z} = xe^z + h'(z)$$

$$\Rightarrow \quad h'(z) = 1$$

$$\Rightarrow \quad h(z) = z + \text{const.}$$

Hence $f(x,y,z) = x(y+e^z) + y + z + C$, where C is a constant, is the required potential function. ##