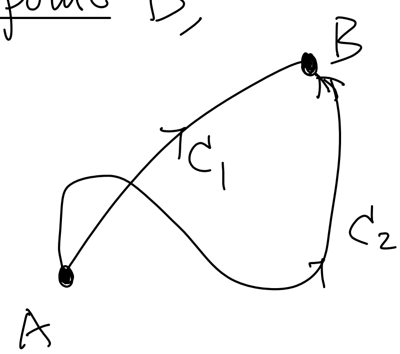
Conservative Vector Field

Def 14: Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be conservative if $\int_C \vec{F}\cdot\hat{T}\,ds \left(= \int_C \vec{F}\cdot d\vec{r}\right)$ along an oriented curve C in Ω depends only on the starting point and end point of C.

Note: This is usually referred as "path independent".

i.e. If C_1 & C_2 are oriented curves with the same starting point A and end point B, then

$$\int_{C_1} \vec{F}\cdot\hat{T}\,ds = \int_{C_2} \vec{F}\cdot\hat{T}\,ds$$



(so the value only depends on the points A & B (& direction))

Notation: If $\vec{F}$ is conservative, we sometimes write

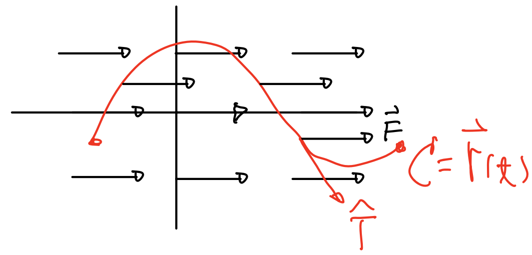
$$\boxed{\int_A^B \vec{F}\cdot\hat{T}\,ds}$$

to denote the common value of $\int_C \vec{F}\cdot\hat{T}\,ds$ along any oriented curve C from A to B.

eg 41: $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$

$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, \quad a \le t \le b$

Then $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$



$$= \int_a^b x'(t)\,dt$$

$$= x(b) - x(a)$$

x-coordinates at $\vec{r}(b)$ & $\vec{r}(a)$ respectively

$\therefore \int_C \vec{F}\cdot\hat{T}\,ds$ depends only on the starting point and the end point

$\Rightarrow \vec{F}$ is conservative.

(Note: $\vec{F} = \vec{\nabla} f$ where $f(x,y) = x$)

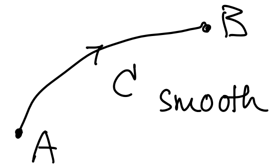
Thm 8 (Fundamental Theorem of Line Integral)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, and $\vec{F} = \vec{\nabla} f$ be the gradient vector field of f. Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = f(B) - f(A)}$$

Pf: Part 1 Assume C is a smooth curve parametrized by

$\vec{r}(t)$, $a \leq t \leq b$

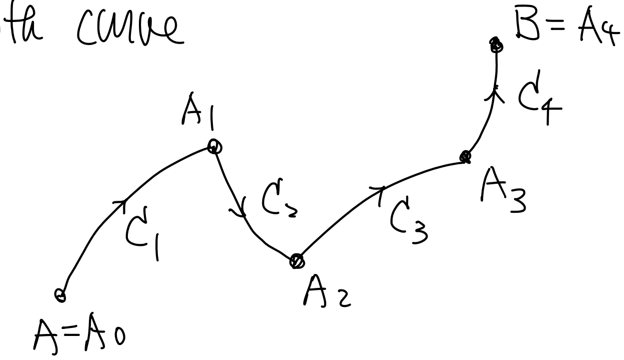


Then
$$\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$
$$= \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$
$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt$$
$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{(Fundamental Thm of Calculus, 1-variable)}$$
$$= f(B) - f(A).$$

Part 2 For a general piecewise smooth curve

$$C = C_1 \cup C_2 \cup \cdots \cup C_k$$

($= C_1 + C_2 + \cdots + C_k$ in order to indicate that they join end-to-end and the orientation of C_i, $i = 1, \cdots, k$, are correct wrt the orientation of C)



where C_i is smooth going from A_{i-1} to A_i

(then $A_0 = A$, $A_k = B$)

Then part 1 implies

$$\int_{C} \vec{F}\cdot\hat{T}\,ds = \int_{\sum_{i=1}^{k} C_i} \vec{F}\cdot\hat{T}\,ds$$

$$= \sum_{i=1}^{k} \int_{C_i} \vec{F}\cdot\hat{T}\,ds \qquad \text{(by def. 9')}$$

$$= \sum_{i=1}^{k} \left[f(A_i) - f(A_{i-1}) \right]$$

$$= f(A_k) - f(A_0)$$

$$= f(B) - f(A) \qquad \#$$

Is the converse of Thm 8 correct? Yes (under a further condition on the domain Ω)

<u>Thm 9</u> Let $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, be open and <u>connected</u>. $\vec{F}$ is a <u>continuous</u> vector field on Ω. Then the following are equivalent.

(a) $\exists$ a C^1 function $f: \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f$$

(b) $\oint_C \vec{F}\cdot d\vec{r} = 0$ along <u>any</u> <u>closed</u> curve C on Ω.

(c) $\vec{F}$ is <u>conservative</u>.

Remarks: (1) The function f in (a) of Thm 9 is called the potential function of $\vec{F}$. It is unique up to an additive constant:

$$\vec{\nabla}(f+c) = \vec{F}, \quad \forall \text{ const. } c.$$

(2) $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} = \vec{\nabla} f \Leftrightarrow Mdx + Ndy + Ldz = df$

(Same for 2-dim)

In this case, $Mdx + Ndy + Ldz$ (or $Mdx + Ndy$ in dim. 2) is called an exact differential form.

Pf: "(a) $\Rightarrow$ (b)"

If f is C^1 and $\vec{F} = \vec{\nabla} f$

and $\vec{r} : [a,b] \to \Omega$ parametrizes C (any closed curve)

C closed $\Longrightarrow \vec{r}(a) = \vec{r}(b) = A$

Fundamental Thm of Line Integral $\Rightarrow$

$$\oint_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(A) - f(A) = 0.$$

(To be cont'd)