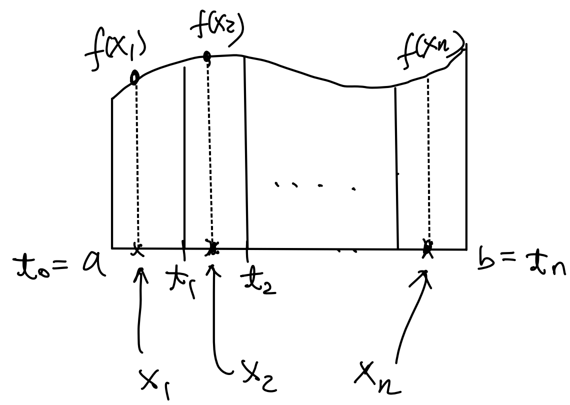
Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH 2060 for rigorous treatment)

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

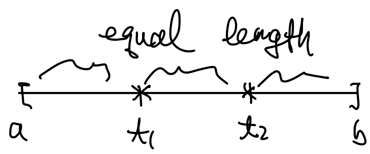
where
- f is a function on the interval $[a,b]$
- P is a partition $a = t_0 < t_1 < t_2 < \cdots < t_n = b$
- $x_k \in [t_{k-1}, t_k]$ and $\Delta x_k = t_k - t_{k-1}$
- $\|P\| = \max_k |\Delta x_k|$



Remark: We usually use uniform partition P

$$a = t_0 < t_1 = a + \frac{1}{n}(b-a) < t_2 = a + \frac{2}{n}(b-a) < \cdots$$

$$\cdots < t_k = a + \frac{k}{n}(b-a) < \cdots = t_n = b$$



In this case, $\|P\| = \max_k |\Delta x_k| = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k) \cdot \frac{b-a}{n}$$

eg1 : Find $\int_0^1 x^3 dx$ (i.e. $f(x)=x^3$ on $[0,1]$)

Soln : (1) One may choose $x_k = \frac{k-1}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$

then $S_n = \sum_{k=1}^{n} f(x_k)\Delta x_k$

$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \cdot \frac{1}{n} \cdots$ (Check)

$= \frac{1}{4}\left(1-\frac{1}{n}\right)^2 \to \frac{1}{4}$ as $n \to \infty$

$\therefore \int_0^1 x^3 dx = \frac{1}{4}$

(2) Or, we can choose $x_k = \frac{k}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$

then $S_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \frac{1}{n} = \cdots = \frac{1}{4}\left(1+\frac{1}{n}\right)^2$

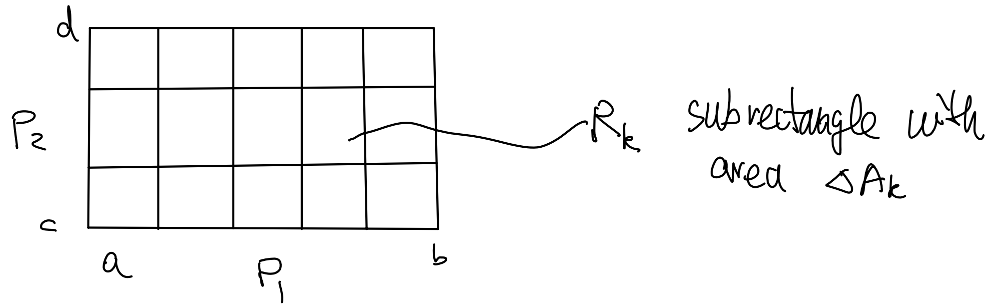
$\to \frac{1}{4}$ as $n \to \infty$ (same limit)

$\therefore \int_0^1 x^3 dx = \frac{1}{4}$.

Remark: We can use any $x_k \in [t_{k-1}, t_k]$ and still get the same $\int_0^1 x^3 dx = \frac{1}{4}$.

This concept can be generalized to any dimension.

For 2-dim., let we first consider a function $f(x,y)$ defined on a rectangle $R = [a,b] \times [c,d] = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ & P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition, subdivision, of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$

Let the sub-rectangles be R_k, $k = 1, \cdots, N$ (= number of sub rectangles)

with areas ΔA_k

Choose point $(x_k, y_k) \in R_k$ (for each $k = 1, \cdots, N$),

then consider the sum

$$S(f, P) = \sum_{k=1}^{N} f(x_k, y_k)\, \Delta A_k$$

<u>Def 1</u>: The function f is said to be <u>integrable</u> over R

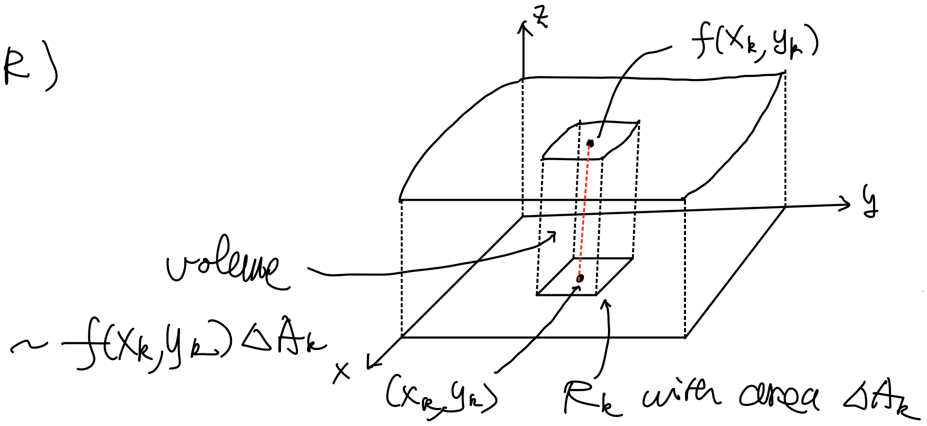
if $$\lim_{\|P\| \to 0} S(f, P) = \lim_{\|P\| \to 0} \sum_{k=1}^{N} f(x_k, y_k)\, \Delta A_k$$

<u>exists</u> and <u>independent</u> of the choice of $(x_k, y_k) \in R_k$.

In this case, the limit is called the <u>(double) integral of f over R</u> and is denoted by

$$\iint_R f(x,y)\, dA \quad \text{or} \quad \iint_R f(x,y)\, dx\, dy$$

Remark: Same as 1-variable, the double integral of f, $f \geq 0$, over R can be interpreted as volume under the graph of f (over R)

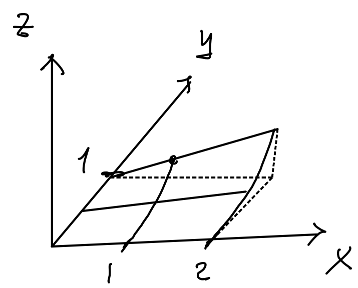


And when $f \equiv 1$,

$$\iint_R 1 \, dA \text{ is the area of } R.$$

eg 2: $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$

(using definition) Find $\iint_R xy^2 \, dxdy$



Soln: Using uniform partitions:

$$P_1 = \{0, \tfrac{2}{n}, \tfrac{4}{n}, \cdots, 2\} \text{ of } [0,2]$$

$$P_2 = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \cdots, 1\} \text{ of } [0,1]$$

$\Rightarrow$ a particular sub-rectangle is

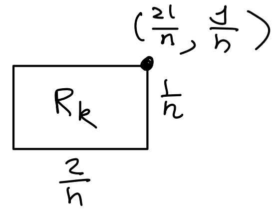
$$R_k = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{(j-1)}{n}, \frac{j}{n}\right] \quad \text{for some } \begin{matrix} i = 1, \cdots, n \\ j = 1, \cdots, n \end{matrix}.$$

(So R_k should be better denoted by R_{ij})

(Assume it is integrable)

One may choose the point

$$(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right) \in R_k$$



and consider the Riemann sum

$$\sum_k f(x_k, y_k)\,\Delta A_k$$

$$= \sum_{i,j=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n}\cdot\frac{1}{n}$$

$$= \frac{4}{n^5} \sum_{i,j=1}^{n} i j^2$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \left[i \sum_{j=1}^{n} j^2 \right]$$

$$= \frac{4}{n^5} \left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j^2\right)$$

$$= \frac{4}{n^5} \frac{n(n+1)}{2} \frac{n(n+1)(2n+1)}{6} \longrightarrow \frac{4\cdot 2}{2\cdot 6} = \frac{2}{3} \quad \text{as } n \to \infty$$

$$\therefore \iint_{[0,2]\times[0,1]} x y^2\,dx\,dy = \frac{2}{3}$$

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Very tedious calculation.

Hence we need the following Theorem:

Thm 1 (Fubini's Theorem (1st form))

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

The last 2 integrals above are called <u>iterated integrals</u>

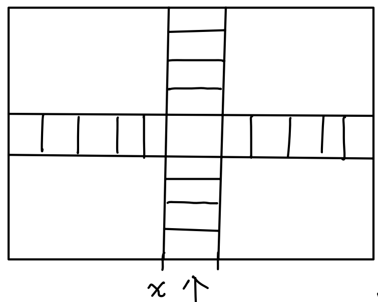
(Pf : Omitted)

<u>Ideas</u>:

sum horizontally first & taking limit →

→ $\int_c^d \int_a^b f(x,y)\,dx\,dy$

(" $\sum_i$ " 1st in eg2)



sum vertically first & taking limit

→ $\int_a^b \int_c^d f(x,y)\,dy\,dx$ (" $\sum_j$ " 1st in eg2)

<u>eg3</u>: Using Fubini to calculate $\iint_R xy^2\,dx\,dy$, $R=[0,2]\times[0,1]$

<u>Soln</u>: By Fubini

$$\iint_R xy^2\,dA = \int_0^2\left(\int_0^1 xy^2\,dy\right)dx$$

$$= \int_0^2\left(x\int_0^1 y^2\,dy\right)dx$$

$$= \int_0^2 \frac{x}{3}\,dx = \frac{2}{3}$$

Check $\int_0^1\left(\int_0^2 xy^2\,dx\right)dy = \frac{2}{3}$.

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(Much easier than eg2)