Math 3280 A 23-11-13

Review.

Let X, Y be independent cts r.v.'s with densities f_X, f_Y respectively, then $X+Y$ has a density given by

$$f_{X+Y}(a) = f_X * f_Y(a)$$
$$= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$
$$= \int_{-\infty}^{\infty} f_X(x) f_Y(a-x)\, dx.$$

- The discrete case: both X and Y are discrete.

We would like to calculate

$$P\{X+Y=a\}.$$

It is easy to see that

$$P\{X+Y=a\} = \sum_x P\{X=x, X+Y=a\}$$
$$= \sum_x P\{X=x, Y=a-x\}$$
$$= \sum_x P\{X=x\} P\{Y=a-x\}$$

Example 1. Let X, Y be independent Poisson r.v.'s with parameters λ_1, λ_2, resp.

Calculate the distribution of $X+Y$.

Solution: $P\{X=k\} = e^{-\lambda_1} \cdot \lambda_1^k / k!$, $k=0,1,2,\cdots$

$P\{Y=k\} = e^{-\lambda_2} \lambda_2^k / k!$, $k=0,1,2,\cdots$

For $n=0,1,2,\cdots$,

$$P\{X+Y=n\} = \sum_{k=0}^{\infty} P\{X=k\} P\{Y=n-k\}$$

(but $P\{Y=n-k\}=0$ if $k>n$)

$$= \sum_{k=0}^{n} P\{X=k\} P\{Y=n-k\}$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\cdot \lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= e^{-\lambda_1 - \lambda_2} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

By the binomial Thm

$$= e^{-\lambda_1 - \lambda_2} (\lambda_1 + \lambda_2)^n / n!$$

(Recall $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.)

Hence $X+Y$ has the Poisson distribution with parameter $\lambda_1 + \lambda_2$.

§ 6.4 Conditional distribution

1. Discrete case.

Def. Let X, Y be two discrete r.v.'s.

Then

$$P\{X = x \mid Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}},$$

provided that $P\{Y = y\} > 0$.

2. Suppose that X and Y are jointly cts with density $f(x,y)$.

Def. The $\boxed{\text{conditional density function of } X}$ given $Y=y$, is given by

$$f_{X|Y}(x|y) := \frac{f(x,y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

Def. For $A \subset \mathbb{R}$, the conditional prob. of X taking values in A given $Y=y$ is given by

$$P\{X \in A \mid Y=y\} = \int_A f_{X|Y}(x|y)\,dx$$

In particular,

$$F_{X|Y}(a|y) := P\{X \le a \mid Y=y\}$$

$$= \int_{-\infty}^{a} f_{X|Y}(x|y)\,dx.$$

Remark: If X and Y are independent,

then $f_{X|Y}(x|y) = f_X(x)$.

(since in such case $f(x, y) = f_X(x) f_Y(y)$

Remark: One may view

$$P\{X \in A \mid Y = y\}$$

$$= \lim_{\varepsilon \to 0} P\{X \in A \mid y - \varepsilon < Y < y + \varepsilon\}$$

$$= \lim_{\varepsilon \to 0} \frac{P\{X \in A, y - \varepsilon < Y < y + \varepsilon\}}{P\{y - \varepsilon < Y < y + \varepsilon\}}$$

Example 2. Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x/y} e^{-y} / y & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X>1 \mid Y=y\}$.

Solution: $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} e^{-x/y} e^{-y} / y \,dx \quad (\text{if } y>0)$$

$$= -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty}$$

$$= e^{-y} \qquad \text{if } y>0.$$

Hence for $y>0$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = e^{-x/y}/y$$

if $x>0$.

Therefore

$$P\{X > 1 \mid Y = y\} = \int_1^\infty e^{-x/y}/y \, dx$$

$$= -e^{-x/y} \Big|_{x=1}^{\infty}$$

$$= e^{-1/y} \qquad \text{if } y > 0.$$