Recall

Axioms of probability (Kolmogorov)

Axiom 1: $0 \le P(E) \le 1$; **Axiom 2:** $P(S) = 1$; **Axiom 3:** For **disjoint** (mutually exclusive) events $(E_n)_{n=1}^{\infty}$, $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$.

From the above axioms, we can deduce the following properties of probability $P(\cdot)$.

Basic properties of $P(\cdot)$

- $P(\emptyset) = 0$
- $P(E^c) = 1 - P(E)$
- (monotone) $P(E) \le P(F)$ if $E \subset F$.
- (inclusion-exclusion) $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{1\le i<j\le n} P(E_iE_j) + \cdots + (-1)^{n+1} P(E_1 \cdots E_n)$
- (finite additive) For disjoint $(E_i)_{i=1}^{n}$, $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$
- (countable subadditive) $P(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} P(E_n)$
- (continuous) Let $(E_n)_{n=1}^{\infty}$ be a sequence of events.

$$\begin{cases} E_n \subset E_{n+1} \implies P(\lim_{n\to\infty} E_n) = P(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n) \\ E_n \supset E_{n+1} \implies P(\lim_{n\to\infty} E_n) = P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n) \end{cases}$$

Basic concepts

Example 1. Roll a die repeatedly until the first 6 appears and then we stop the experiment.

(a) What's the sample space?

(b) Let $n \ge 1$. Explicitly describe the event E_n that we roll the die for $\le n$ times and stop.

(c) What's the event $(\bigcup_{n=1}^{\infty} E_n)^c$?

Solution. (a) The sample space

$$S = \bigcup_{k=0}^{\infty} \Big\{ (i_1, \dots, i_k, 6) \colon i_1, \dots i_k \in \{1, \dots 5\} \Big\} \cup \Big\{ (i_k)_{k=1}^{\infty} \colon \forall\, k \in \mathbb{N}, i_k \in \{1, \dots 5\} \Big\}$$

with convention $\{(i_1, i_0, 6)\} = \{(6)\}$. The first part represents the outcomes that we stop the experiment after rolling finite times while the last set consists of the outcomes that we never stop.

(b) Similarly, $E_n = \bigcup_{k=1}^{n-1} \Big\{ (i_1, \dots, i_k, 6) \colon i_1, \dots i_k \in \{1, \dots 5\} \Big\} \cup \Big\{ (6) \Big\}$.

(c) From the expressions of S and E_n, we have

$$\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \left\{(i_k)_{k=1}^{\infty} : \forall\, k \in \mathbb{N}, i_k \in \{1, \ldots 5\}\right\}$$

which is the event that we never stop the experiment. □

Sample spaces with equally likely outcomes

Example 2. Roll a die twice. What's the probability that the second number is larger than the first?

Solution. Explicitly write down the sample space $S = \left\{(i, j) : i, j \in \{1, \ldots, 6\}\right\}$ and the event $E = \left\{(i, j) : i < j\right\} = \{(1, 2), \ldots (1, 5), \ldots, (5, 6)\}$. Then $|S| = 36$ and $|E| = 5 + 4 + \cdots + 1 = 15$. By the assumption on equal probabilities, we have $P(E) = 15/36 = 5/12$. □

Example 3. In a game, the total 52 (4×13) cards are dealt out to 4 players. What's the probability of

(a) the event A that one of the players receives all 13 heart ♡ cards?

(b) the event B that each player receives 1 ace?

(c) the event C that each player receives at least 1 heart ♡ cards?

Solution. (a) Let $E_i, (i = 1, \ldots, 4)$ be the event that the player i receives 13 hearts. Then $P(E_i) = 1/\binom{52}{13}$ which can the obtained by reasoning: choose 13 cards for the player i from 52 cards, only 1 selection consists of all heart cards.

Since there are only 13 heart cards, any two players can not have all heart cards at the same time, i.e., E_i are disjoint. By finite additivity,

$$P(A) = P\left(\bigcup_{i=1}^{4} E_i\right) = \sum_{i=1}^{4} P(E_i) = \frac{4}{\binom{52}{13}} \approx 6.3 \times 10^{-12}.$$

(b) There are $\binom{52}{13,13,13,13}$ ways of dealing out 52 cards to 4 players with equal probabilities. To determine the outcomes making event B happen, we first determine the positions of 4 aces which results in 4! permutations, then we count the ways to distribute the remaining $52 - 4 = 48$ cards to the 4 players. Hence

$$P(B) = \frac{4! \times \binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} \approx 0.1055.$$

(c) Let $C_i (i = 1, \ldots, 4)$ denote the event that the player i does not receive heart cards. By taking the complement,

$$P(C) = 1 - P\left(\bigcup_{i=1}^{4} C_i\right).$$

To obtain $P\left(\bigcup_{i=1}^4 C_i\right)$, we will use the inclusion-exclusion principle. It follows from the similar arguments of (a) that

$$
\begin{aligned}
&i = 1, \ldots 4 && P(C_i) = \frac{\binom{39}{13}}{\binom{52}{13}} \\
&1 \le i < j \le 4 && P(C_i C_j) = \frac{\binom{39}{26}}{\binom{52}{26}} \\
&1 \le i < j < k \le 4 && P(C_i C_j C_k) = \frac{\binom{39}{39}}{\binom{52}{39}}.
\end{aligned}
$$

Notice $P(C_1C_2C_3C_4) = 0$ since the 4 players cannot avoid having heard cards at the same time. Hence by inclusion-exclusion principle,

$$P(\bigcup_{i=1}^4 C_i) = 4 \times \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \times \frac{\binom{39}{26}}{\binom{52}{26}} + \binom{4}{3} \times \frac{\binom{39}{39}}{\binom{52}{39}} - 0.$$

Thus $P(C) = 1 - P(\bigcup_{i=1}^4 C_i) \approx 0.9488$.

□