Recall

For $n \in \mathbb{N}$, let $X, \widetilde{X}, Y$ and X_i, $i = 1, \ldots, n$ be random variables.

Properties of expectation

- Let $g\colon \mathbb{R}^2 \to \mathbb{R}$.
  - If X, Y discrete with joint PMF $p(x, y)$, then
  $$E[g(X,Y)] = \sum_x \sum_y g(x,y)p(x,y).$$
  - If X, Y joint continuous with joint PDF $f(x, y)$, then
  $$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy.$$
- (linearity) $\forall\, \alpha, \beta \in \mathbb{R}$, $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$. By induction, $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i]$.
  In general, to obtain $E[\sum_{i=1}^\infty X_i] = \sum_{i=1}^\infty E[X_i]$, we need some additional conditions. Two of such conditions are: (1) $X_i \geq 0$ for all $i \in \mathbb{N}$. **Or** (2) $\sum_{i=1}^\infty E[|X_i|] < \infty$.
- (monotone) If $X \leq Y$, then $E[X] \leq E[Y]$. In particular, $|E[X]| \leq E[|X|]$.

Covariance

- $\operatorname{Cov}(X, Y) := E\left[(X - E[X])(Y - E[Y])\right] = E[XY] - E[X]E[Y]$.
- X, Y independent $\implies E[XY] = E[X]E[Y] \iff \operatorname{Cov}(X, Y) = 0$.
- Properties of $\operatorname{Cov}(\cdot, \cdot)$:
  (1) $\operatorname{Cov}(X, X) = \operatorname{Var}(X) \geq 0$.
  (2) $\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$.
  (3) $\operatorname{Cov}(\alpha X + \widetilde{X}, Y) = \alpha \operatorname{Cov}(X, Y) + \operatorname{Cov}(\widetilde{X}, Y)$ for $\alpha \in \mathbb{R}$.

  It follows from (2) and (3) that $\operatorname{Cov}(\cdot, \cdot)$ is bilinear.
- $\operatorname{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \operatorname{Var}(X_i) + 2\sum_{1\leq i<j\leq n} \operatorname{Cov}(X_i, X_j)$. In particular, $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y)$.
  Moreover, if $X_1, \ldots, X_n$ are pairwise independent, then $\operatorname{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \operatorname{Var}(X_i)$.

Examples

Example 1. For $L > 0$, let $X, Y \overset{i.i.d.}{\sim} U(0, L)$. Find $E[|X - Y|]$.

Solution. By independence, the joint PDF of X, Y is

$$f(x,y) = f_X(x)f_Y(y) = \frac{1}{L}\chi_{(0,L)}(x)\frac{1}{L}\chi_{(0,L)}(y) = \frac{1}{L^2}\chi_{(0,L)\times(0,L)}(x,y).$$

Applying the formula of $E[g(X,Y)]$, we have

$$\begin{aligned} E[|X-Y|] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|f(x,y)dxdy \\ &= \int_0^L\int_0^x (x-y)\frac{1}{L^2}dydx + \int_0^L\int_0^y (y-x)\frac{1}{L^2}dxdy \\ &= \frac{1}{L^2}\int_0^L x^2 dx = \frac{L}{3}. \end{aligned}$$

□

Denote the sample space by Ω. Let A, B be events $\subset \Omega$. Let χ_A be the *indicator variable* with respect to A, that is, for $\omega \in \Omega$,

$$\chi_A(\omega) := \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$$

Then the following conclusions can be checked:

(a) $E[\chi_A] = P(A)$.

(b) $\chi_{A^c} = 1 - \chi_A$; $\chi_{A\cap B} = \chi_A\chi_B$.

Example 2. In the above notation, prove $\text{Cov}(\chi_A, \chi_B) = P(B)(P(A|B) - P(A))$ if $P(B) > 0$.

Proof. By (a) and (b),

$$\begin{aligned} \text{Cov}(\chi_A,\chi_B) &= E[\chi_A\chi_B] - E[\chi_A]E[\chi_B] \\ &= E[\chi_{A\cap B}] - P(A)P(B) \\ &= P(A\cap B) - P(A)P(B) \\ &= P(B)(P(A|B) - P(A)). \end{aligned}$$

□

Remark. By Example 2, the events A and B are independent $\iff$ $\text{Cov}(\chi_A, \chi_B) = 0$.

Example 3. Let X, Y be random variables with joint PDF

$$f(x,y) = \begin{cases} \frac{1}{y}e^{-y-x/y} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\text{Cov}(X, Y) = 1$.

Solution. By change of variable, let $t = x/y$ in the inner integral (where y is fixed) when it appears during the computations below. Then $dx = y\,dt$. By applying the formula for $E[g(X,Y)]$,

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = \int_0^{\infty}\int_0^{\infty} \frac{x}{y}e^{-y-x/y}dxdy = \int_0^{\infty}\int_0^{\infty} te^{-y-t}ydtdy \\ &= \int_0^{\infty} ye^{-y}dy \int_0^{\infty} te^{-t}dt = 1\times 1 = 1\end{aligned}$$

and

$$\begin{aligned}E[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy = \int_0^{\infty}\int_0^{\infty} e^{-y-x/y}dxdy = \int_0^{\infty}\int_0^{\infty} e^{-y-t}ydtdy \\ &= \int_0^{\infty} ye^{-y}dy \int_0^{\infty} e^{-t}dt = 1\times 1 = 1.\end{aligned}$$

Also,

$$\begin{aligned}E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dx = \int_0^{\infty}\int_0^{\infty} xe^{-y-x/y}dxdy = \int_0^{\infty}\int_0^{\infty} yte^{-y-t}ydtdy \\ &= \int_0^{\infty} y^2e^{-y}dy \int_0^{\infty} te^{-t}dt = 2\times 1 = 2.\end{aligned}$$

Hence $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 2 - 1\times 1 = 1.$ □