THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2023-2024 Term 1
Suggested Solutions of Midterm Exam

Q1 (10pts)

Suppose that E, F, G are independent events in a probability space. Let E^c and F^c denote the complements of E and F, respectively.

(a) Show that E^c and F^c are independent.

Since

$$\begin{aligned}
P(E^c \cap F^c) &= P\left((E \cup F)^c\right) \\
&= 1 - P(E \cup F) \\
&= 1 - P(E) - P(F) + P(E \cap F) \quad \text{(Inclusion-exclusion identity)} \\
&= 1 - P(E) - P(F) + P(E)P(F) \quad (E \text{ and } F \text{ are independent}) \\
&= (1 - P(E))\,(1 - P(F)) \\
&= P(E^c)P(F^c),
\end{aligned}$$

then by the definition, E^c and F^c and independent.

(b) Is $E \cup F$ independent of G? Justify your answer.

Since

$$\begin{aligned}
P\left((E \cup F) \cap G\right) &= P\left((E \cap G) \cup (F \cap G)\right) \\
&\quad \text{(Inclusion-exclusion identity)} \\
&= P(E \cap G) + P(F \cap G) - P\left((E \cap G) \cap (F \cap G)\right) \\
&= P(E \cap G) + P(F \cap G) - P(E \cap F \cap G) \\
&\quad (E,\ F,\ G \text{ are independent}) \\
&= P(E)P(G) + P(F)P(G) - P(E)P(F)P(G) \\
&= \left[P(E) + P(F) - P(E)P(F)\right] P(G) \\
&= \left[P(E) + P(F) - P(E \cap F)\right] P(G) \\
&= P(E \cup F)P(G),
\end{aligned}$$

then by the definition, $E \cup F$ and G are independent.

Q2 (15pts)

Suppose events A, B, and C are independent with probabilities 1/2, 1/6, and 1/3 respectively. Calculate the following probabilities:

(a) $P(A \cap B \cap C)$.

Since A, B, C are independent, then

$$P(A \cap B \cap C) = P(A)P(B)P(C) = \frac{1}{2} \times \frac{1}{6} \times \frac{1}{3} = \frac{1}{36}.$$

(b) $P(A \cup B \cup C)$.

By the Inclusion-exclusion identity,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) \\ &- P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C) \\ &= \frac{1}{2} + \frac{1}{6} + \frac{1}{3} - \frac{1}{2} \times \frac{1}{6} - \frac{1}{2} \times \frac{1}{3} - \frac{1}{6} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{6} \times \frac{1}{3} \\ &= \frac{13}{18}. \end{aligned}$$

(c) $P(\text{exactly one of the three events occurs})$.

Note that the event $E := \{\text{exactly one of the three events occurs}\}$ can be written as

$$[(A \cup B \cup C) \setminus (B \cup C)] \cup [(A \cup B \cup C) \setminus (A \cup C)] \cup [(A \cup B \cup C) \setminus (A \cup B)].$$

Because A, B, C are independent, then

$$P(B \cup C) = P(B) + P(C) - P(B)P(C) = \frac{4}{9},$$

$$P(A \cup C) = P(A) + P(C) - P(A)P(C) = \frac{2}{3},$$

$$P(A \cup B) = P(A) + P(B) - P(A)P(B) = \frac{7}{12}.$$

Since the events $(A \cup B \cup C) \setminus (B \cup C)$, $(A \cup B \cup C) \setminus (A \cup C)$ and $(A \cup B \cup C) \setminus (A \cup B)$ are disjoint, then

$$\begin{aligned}
P(E) &= P((A \cup B \cup C) \setminus (B \cup C)) \\
&\quad + P((A \cup B \cup C) \setminus (A \cup C)) + P((A \cup B \cup C) \setminus (A \cup B)) \\
&= P(A \cup B \cup C) - P(B \cup C) \\
&\quad + P(A \cup B \cup C) - P(A \cup C) + P(A \cup B \cup C) - P(A \cup B) \\
&= \frac{13}{18} - \frac{4}{9} + \frac{13}{18} - \frac{2}{3} + \frac{13}{18} - \frac{7}{12} \\
&= \frac{17}{36}.
\end{aligned}$$

Q3 (10pts)

Two cards are randomly chosen, without replacement, from an ordinary deck of 52 playing cards. Compute the probability that they have the different values.

Let A be the event that the chosen two cards have the different values. And let S be the sample space.

First, we need to choose two different values from the total 13 values. There are $\binom{13}{2}$ possible choices. Denote the two different values by a and b. Second, since there are four cards of each value, then in order to choose one card of value a and one card of value b, there are 4^2 possible choices. It follows that $|A| = \binom{13}{2} \times 4^2$.

Therefore,

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{13}{2} \times 4^2}{\binom{52}{2}} = \frac{16}{17}.$$

Q4 (16pts)

A coin having probability p of landing on heads is tossed repeatedly until it comes up to the second head. Let X denote the numbers of times we have to toos the coin until it comes up the second head.

(a) Calculate $P(X = 2)$ and $P(X = 3)$.

Use H to denote the head, and T to denote the tail. Then

$$P(X=2)=P\{(H,H)\}=p\times p=p^2,$$

$$P(X=3)=P\{(H,T,H),(T,H,H)\}=p\cdot(1-p)\cdot p+(1-p)\cdot p\cdot p=2p^2(1-p).$$

(b) Calculate $P(X=n)$ for integers $n\geq 2$.

Note that the event $\{X=n\}$ it equivalent to the event that the n-th trial is head, and there are $(n-2)$ tails and one head among the first $(n-1)$ trials. Since there are $\binom{n-1}{1}$ possible choices to choose the time that the coin shows up head among the first $(n-1)$ trials, and by the independence,

$$P(X=n)=\binom{n-1}{1}p\cdot(1-p)^{n-2}\cdot p=(n-1)p^2(1-p)^{n-2}.$$

Q5 (14pts)

Let Z be a standard normal random variable.

(a) Find the probability density function of $X=3Z$.

$$E[X]=3E[Z]=0, \mathrm{Var}(X)=3^2\mathrm{Var}(Z)=3^2,$$

we have $X\sim N\left(0,3^2\right)$. Then

$$f(x)=\frac{1}{3\sqrt{2\pi}}e^{-\frac{x^2}{18}},\ x\in\mathbb{R}.$$

(b) Find $E[Y]$ for $Y=Z^2$.

$$E[Y]=E[Z^2]=\mathrm{Var}(Z)+E[Z]^2=1.$$

Q6 (10pts)

Let X be a Poisson random variable with parameter λ. Calculate the conditional probability $P\{X = 2|X > 1\}$.

$$\begin{aligned} P\{X = 2|X > 1\} &= \frac{P\{X = 2, X > 1\}}{P\{X > 1\}} \\ &= \frac{P\{X = 2\}}{1 - P\{X = 1\} - P\{X = 0\}} \\ &= \frac{\frac{\lambda^2}{2} e^{-\lambda}}{1 - \lambda e^{-\lambda} - e^{-\lambda}} \\ &= \frac{\lambda^2 e^{-\lambda}}{2 - 2\lambda e^{-\lambda} - 2e^{-\lambda}}. \end{aligned}$$

Q7 (10pts)

Box A contains 2 black and 5 white balls, whereas box B contains 1 black and 1 white ball. A ball is randomly chosen from box A and transferred to box B. Then a ball is randomly selected from box B. What is the probability that the ball selected from box B is black?

Denote the event "the ball chosen from box A is black" by A_b, and A_b^c its complement. Denote the event "the ball chosen from box B is black after a ball randomly chosen from box A is transferred to box B" by B_b. Then by law of total probability,

$$\begin{aligned} P(B_b) &= P(B_b|A_b)P(A_b) + P(B_b|A_b^c)P(A_b^c) \\ &= \frac{2}{3} \times \frac{2}{7} + \frac{1}{3} \times \frac{5}{7} \\ &= \frac{3}{7}. \end{aligned}$$

Q8 (15pts)

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} xe^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P\{1 \leq X \leq 2\}$;

(b) Find $E[X]$;

(c) Find $E[X^{-1}]$.

(a) By integration by parts,

$$\begin{aligned} P\{1 \leq X \leq 2\} &= \int_1^2 xe^{-x}dx \\ &= [-xe^{-x}]_1^2 + \int_1^2 e^{-x}dx \\ &= -2e^{-2} + e^{-1} + [-e^{-x}]_1^2 \\ &= 2e^{-1} - 3e^{-2}. \end{aligned}$$

(b) By integration by parts,

$$\begin{aligned} E[X] &= \int_0^{+\infty} xf(x)dx \\ &= \int_0^{+\infty} x^2e^{-x}dx \\ &= [-x^2e^{-x}]_0^{+\infty} + 2\int_0^{+\infty} xe^{-x}dx \\ &= 2[-xe^{-x}]_0^{+\infty} + 2\int_0^{\infty} e^{-x}dx \\ &= 2. \end{aligned}$$

(c)

$$
\begin{aligned}
E[X^{-1}] &= \int_0^{+\infty} x^{-1} f(x) dx \\
&= \int_0^{+\infty} e^{-x} dx \\
&= [-e^{-x}]_0^{+\infty} \\
&= 1.
\end{aligned}
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