CHINESE UNIVERSITY OF HONG KONG

MATH2010E
Advanced Calculus I

香 港 中 文 大 學
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Instructor: LUKE PEACHEY

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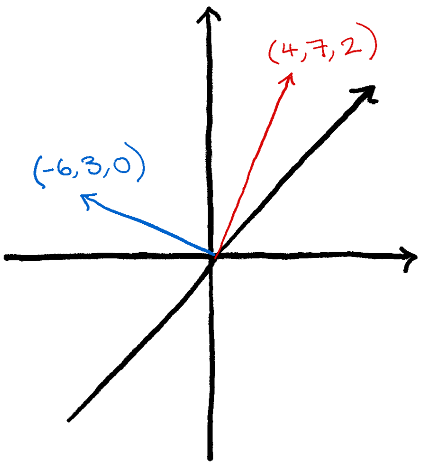
Contents

Week 1

1.1 Euclidean Space

In this course, we are concerned with the analysis of functions between finite dimensional vector spaces over the real numbers $\mathbb{R}$. For some fixed dimension $n \in \mathbb{N}$, up to (linear) isomorphism such a vector space is n-dimensional Euclidean space $\mathbb{R}^n$.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n-times} = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, \quad \forall i \in \{1, \ldots, n\}\}.$$



Notation: In these printed notes we will use regular lettering $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to denote a vector, with subscripts denoting the component of such a vector in Cartesian coordinates. In some cases, we will also use capital letters $A, B, C, \ldots$ to represents points within $\mathbb{R}^n$, and use $\overrightarrow{AB}$ to denote the vector starting from point A and ending at point B.

Basic Operations on Vectors

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be a pair of vectors and $\lambda \in \mathbb{R}$ a scalar.

- Equality: $x = y \iff x_i = y_i$, for all $i \in \{1, \ldots, n\}$.

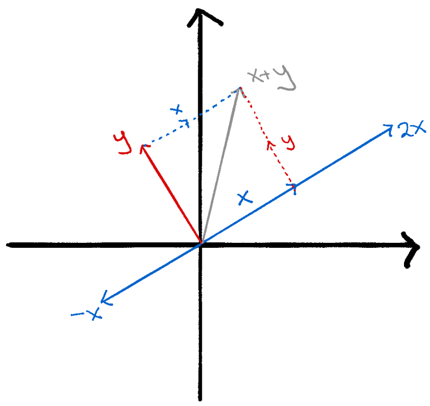
  'Two vectors are the same if and only if all of their components agree.'

- Addition: $x + y = (x_1 + y_1, \ldots, x_n + y_n)$.

  'Vectors are summed component wise.'

- Scalar multiplication: $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$.

  'Scaling a vector scales all of its components by the same amount.'



Length and the Dot Product

As well as the usual vector space structure of $\mathbb{R}^n$, we also equip it with its natural inner product, known as the *dot product*.

Definition 1.1. *For any pair of vectors $x, y \in \mathbb{R}^n$ we define their dot product to be*

$$x \cdot y := \sum_{i=1}^{n} x_i y_i \in \mathbb{R}.$$

For any vector $x \in \mathbb{R}^n$ we define its (Euclidean) length $\|x\|$ via the equation

$$\|x\|^2 := x \cdot x = x_1^2 + \cdots + x_n^2.$$

We note that the dot product is a map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which takes a pair of vectors and returns a scalar value.

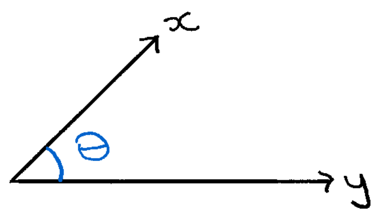
Remark. *In dimension $n \leq 3$, $\|x\|$ agrees with the usual notion of length you calculate by applying the Pythagorean theorem.*

Lemma 1. *Let $x, y, z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The dot product enjoys the following properties.*

a) $x \cdot y = y \cdot x$.

b) $(x + y) \cdot z = x \cdot z + y \cdot z$.

c) $(\lambda x) \cdot y = \lambda(x \cdot y) = x \cdot (\lambda y)$.



d) $x \cdot x \geq 0$, with equality iff $x = 0 \in \mathbb{R}^n$.

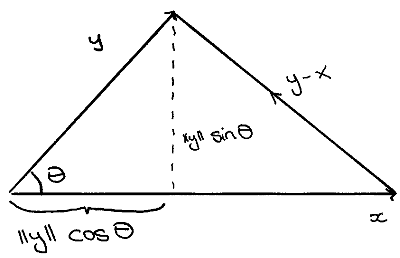
e) If $x \neq 0$ and $y \neq 0$, then $x \cdot y = ||x|| ||y|| \cos \theta$,

where θ is the size of the angle formed between the vectors x and y.

Proof. Properties a)-d) follow trivialy from the definitions and are left as an exercise. To show e) we first consider the case where $\theta \in [0, \frac{\pi}{2}]$. By the definition of the dot product, we have

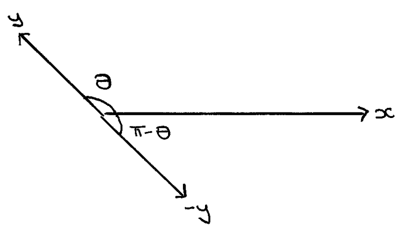
$$\begin{aligned} ||y - x||^2 &= (y - x) \cdot (y - x) \\ &= y \cdot y - x \cdot y - y \cdot x + x \cdot x \\ &= ||y||^2 + ||x||^2 - 2x \cdot y. \end{aligned}$$

Alternatively, by the Pythagorean theorem



$$\begin{aligned} ||y - x||^2 &= (||x|| - ||y|| \cos \theta)^2 + (||y|| \sin \theta)^2 \\ &= ||y||^2 + ||x||^2 - 2||x|| ||y|| \cos \theta. \end{aligned}$$

Comparing the two gives the result for $\theta \in [0, \frac{\pi}{2}]$. Finally, if $\theta \in (\frac{\pi}{2}, \pi]$, we consider instead the



vectors x and $-y$, which now form an angle of size $\pi - \theta \in [0, \frac{\pi}{2})$. We may then apply the above result to conclude that

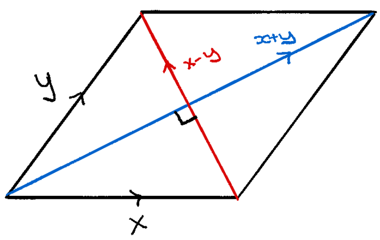
$$x \cdot (-y) = \|x\|\|-y\| \cos(\pi - \theta).$$

Since $x \cdot (-y) = -(x \cdot y)$, $\|-y\| = \|y\|$, and $\cos(\pi - \theta) = -\cos(\theta)$, the result follows. $\square$

Remark. *For any two vectors $x, y \in \mathbb{R}^n$ which form an angle of size θ,*

$$x \text{ and } y \text{ are orthogonal} \iff \theta = \frac{\pi}{2} \iff \cos(\theta) = 0 \iff x \cdot y = 0.$$

The following is a simple lemma which states that the diagonals of a parallelogram are orthogonal to each other if and only if the parallelogram is in fact a rhombus.



Lemma 2. *For any two vectors $x, y \in \mathbb{R}^n$, $x + y$ is orthogonal to $x - y$ if and only if x and y have the same length.*

Proof. From the previous remark, $x + y$ and $x - y$ are orthogonal iff $(x + y) \cdot (x - y) = 0$, but a direct calculation leads to

$$(x + y) \cdot (x - y) = x \cdot x + x \cdot y - x \cdot y - y \cdot y = \|x\|^2 - \|y\|^2,$$

from which the result follows immediately. $\square$

As an application of the previous lemma, we give a simple proof that for any triangle lying on a circle with one edge given by a diameter, the opposite angle is $\frac{\pi}{2}$.

If A, B, C denote the vertices of the triangle with A and B lying on the diameter and O denoting the centre of the circle, then $\overrightarrow{AO} = -\overrightarrow{BO}$, and hence

$$\overrightarrow{AC} = \overrightarrow{OC} + \overrightarrow{AO}, \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{AO}.$$

Since the lengths of $\overrightarrow{OC}$ and $\overrightarrow{AO}$ are the same (the radius of the circle), we can apply the previous lemma to conclude that $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are orthogonal.

Cauchy-Schwarz and the Triangle Inequality.

The following is an important inequality which bounds the size of the dot product by the product of the lengths.

Lemma 3 (Cauchy-Schwarz inequality)**.** *For any $x, y \in \mathbb{R}^n$,*

$$|x \cdot y| \leq \|x\|\|y\|. \tag{1.1}$$

Moreover, equality holds if and only if x and y are parallel. That is, $x = \lambda y$ or $y = \lambda x$ for some $\lambda \in \mathbb{R}$.

Remark. *For $n \leq 3$, this follows immediately from the equality $x \cdot y = \|x\|\|y\| \cos\theta$.*

Proof. If $y = 0 \in \mathbb{R}^n$, then the result is trivially true with $y = 0 \cdot x$. Therefore, we may assume throughout the proof that $y \neq 0$. For any $\lambda \in \mathbb{R}$, we consider the quantity

$$0 \leq \|x - \lambda y\|^2 := (x - \lambda y) \cdot (x - \lambda y) = \|x\|^2 - 2\lambda x \cdot y + \lambda^2 \|y\|^2. \tag{1.2}$$

As $y \neq 0$, we may choose $\lambda := \frac{x \cdot y}{\|y\|^2} \in \mathbb{R}$ in (1.2), which simplifies to

$$\|x\|^2 - \frac{|x \cdot y|}{\|y\|^2} \geq 0.$$

Rearranging, we can conclude (1.1) holds. Moreover, equality in (1.1) holds iff $\|x - \lambda y\| = 0$ for our choice of λ, which happens iff x and y are parallel. □

Remark. *Let $x, y \in \mathbb{R}^n$ be non-zero vectors. For $n \leq 3$, we proved that*

$$x \cdot y = \|x\|\|y\| \cos\theta.$$

Rearranging, we have

$$\theta = \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right) \in [0, \pi].$$

For any $n \in \mathbb{N}$, Cauchy-Schwarz implies that

$$-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1.$$

Thus, we can define

$$\theta = \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right) \in [0, \pi],$$

to be the angle between x and y in all dimensions.

Example 4. For the pair of vectors $x = (1, 5, 2, 3, 4), y = (3, 3, 1, 1, 0) \in \mathbb{R}^5$, we have

$$x \cdot x = 55, \quad y \cdot y = 20, \quad x \cdot y = 23.$$

Therefore, the angle between them is $\theta = \arccos\left(\frac{2.3}{\sqrt{11}}\right)$.

Just like for the absolute value in $\mathbb{R}$, the triangle inequality holds for the length of vectors in $\mathbb{R}^n$ as well.

Lemma 5. *For any $x, y \in \mathbb{R}^n$,*

$$\|x + y\| \leq \|x\| + \|y\|. \tag{1.3}$$

Moreover, equality holds if and only if $x = \lambda y$ or $y = \lambda x$ for some $\lambda \geq 0$.

Proof. By expanding the dot product and using the Cauchy-Schwarz inequality

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2x \cdot y + \|y\|^2 \\ &\leq \|x\|^2 + 2\,|x \cdot y| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

which is precisely (1.3). Moreover, we have equality in (1.3) iff we have equality in Cauchy-Schwarz and $x \cdot y = |x \cdot y|$, which happens iff x is parallel to y with $\lambda \geq 0$. □

The Cross Product

For this section, we restrict our attention to the case $n = 3$.

Before stating the definition of the cross product of two vectors in $\mathbb{R}^3$, we recall that the determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and for a 3×3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Definition 1.2. *For any two vectors $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we define thie cross product to be*

$$x \times y := \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \hat{i} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \hat{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \hat{k} \in \mathbb{R}^3, \tag{1.4}$$

where $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$.

Example 6. If $x = (2, 3, 4)$ and $y = (1, 2, 3)$, then

$$\begin{aligned} x \times y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} &= \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k} \\ &= \hat{i} - 2\hat{j} + \hat{k} \\ &= (1, -2, 1). \end{aligned}$$

Remark.
$$\begin{aligned} \hat{\imath}\times\hat{\imath} &= 0, & \hat{\imath}\times\hat{\jmath} &= \hat{k}, & \hat{\imath}\times\hat{k} &= -\hat{\jmath}, \\ \hat{\jmath}\times\hat{\imath} &= -\hat{k}, & \hat{\jmath}\times\hat{\jmath} &= 0, & \hat{\jmath}\times\hat{k} &= \hat{\imath}, \\ \hat{k}\times\hat{\imath} &= \hat{\jmath}, & \hat{k}\times\hat{\jmath} &= -\hat{\imath}, & \hat{k}\times\hat{k} &= 0. \end{aligned}$$

The cross product satisfies the ***right hand rule****.*

Properties of the Cross Product

Let $x, y, z \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$.

a) $x \times y = -y \times x$

b) $(\alpha x + \beta y) \times z = \alpha(x \times z) + \beta(y \times z)$.

c) $(x \times y) \cdot x = (x \times y) \cdot y = 0$, so $x \times y$ is orthogonal to the plane spanned by x and y.

d) If θ denotes the angle between x and y, then

$$\|x \times y\| = \|x\|\|y\| \sin\theta,$$

which is equal to the area of the parallelogram generated by x and y.

Proof. Properties a)-c) follow follow directly from the definition and are left as an Exercise. To show d) we begin by expanding the term $(x \times y) \cdot (x \times y)$, and find that

$$\begin{aligned} \|x \times y\|^2 &= \|x\|^2\|y\|^2 - (x \cdot y)^2 \\ &= \|x\|^2\|y\|^2(1 - \cos^2(\theta)) \\ &= \|x\|^2\|y\|^2 \sin^2(\theta). \end{aligned}$$

Note that the area is given by the base $\|x\|$ times the height $\|y\| \sin\theta$. $\square$

Remark. *In the case that $x, y \in \mathbb{R}^2$, we see that*

$$x \times y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & x_2 & 0 \\ y_1 & y_2 & 0 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \hat{k}, \tag{1.5}$$

and so $\|x \times y\| = x_1 y_2 - x_2 y_1$.

Triple Product

For any $x, y, z \in \mathbb{R}^3$, we define their triple product as $(x \times y) \cdot z \in \mathbb{R}$.

Unravelling the definition, we see that

$$(x \times y) \cdot z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \tag{1.6}$$

It follows from (1.6) that

$$(x \times y) \cdot z = (y \times z) \cdot x = (z \times x) \cdot y = -(y \times x) \cdot z = -(z \times y) \cdot x = -(x \times z) \cdot y$$

The triple product has the following geometric interpretation.

Lemma 7. $|(x \times y) \cdot z|$ *is the volume of the parallelepiped spanned by* x, y *and* z.

Proof. Let α denote the angle formed between the vector $x \times y$ and z.

Note that, after possibly replacing $x \times y$ with $y \times x$, we may assume that $\alpha \in [0, \frac{\pi}{2}]$.

Since $(x \times y) \cdot z = \|x \times y\| \|z\| \cos \alpha$, and our parallelepiped has base area $\|x \times y\|$ and height $\|z\| \cos \alpha$, the result follows. □

Remark. *The triple product* $(x \times y) \cdot z = 0$ *iff the volume of the parallelepiped vanishes iff* $\{x, y, z\}$ *are linearly dependent.*

Example 8. Consider the parallelepiped spanned by the vectors $(1, 1, 0), (0, 1, 0), (1, 1, 1) \in \mathbb{R}^3$. The triple product of these three vectors is

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1,$$

and therefore the parallelepiped has unit volume.

Linear Dependence

In this subsection, we define what it means for a collection of vectors to be linearly independent.

Definition 1.3. *Let* $v_1, \ldots, v_k \in \mathbb{R}^n$ *be a collection of vectors. We say that these vectors are linearly dependent if there exists a non-zero vector* $\alpha \in \mathbb{R}^k$ *such that*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0.$$

Otherwise, we say that the vectors are linearly independent.

Another way to state this definition is that for linearly independent $v_1, \ldots, v_k \in \mathbb{R}^n$, the only scalars $\alpha_i \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0,$$

are $\alpha_i = 0$, for all $i \in \{1, \ldots, k\}$.

Example 9. For two vectors $v_1, v_2 \in \mathbb{R}^n$, v_1 and v_2 are linearly dependent if

$$\alpha_1 v_1 = -\alpha_2 v_2,$$

for some non-zero $\alpha \in \mathbb{R}^2$. Without loss of generality, we may assume that $\alpha_1 \neq 0$, and so dividing through by α_1 we have

$$v_1 = \frac{-\alpha_2}{\alpha_1} v_2,$$

so v_1 and v_2 are parallel.

Example 10. Consider the three vectors $(0, 2, 1), (1, -4, -1), (1, 0, 1) \in \mathbb{R}^3$. We note that

$$2 \cdot (0, 2, 1) + (1, -4, -1) - (1, 0, 1) = 0,$$

and so these three vectors are linearly dependent. In particular, these three vectors all lie in the same plane.

1.2 Affine Subspaces

We begin with the following standard definition of linear subspaces of $\mathbb{R}^n$.

Definition 1.4. *A non-empty subset $X \subseteq \mathbb{R}^n$ is a linear subspace if*

$$\lambda x + \mu y \in X, \quad \forall x, y \in X, \quad \forall \lambda, \mu \in \mathbb{R}.$$

Example 11 (Examples in $\mathbb{R}^3$)**.** The trivial subspace $X = \{0\}$ and the entire space itself $\mathbb{R}^3$ are the easiest examples of linear subspaces in $\mathbb{R}^3$.

The x-axis $\{(\lambda, 0, 0) : \lambda \in \mathbb{R}\}$ is certainly a linear subspace. In fact, for any vector $v \in \mathbb{R}^3$, the set

$$X := \{\lambda v : \lambda \in \mathbb{R}\},$$

is a linear subspace. It is a line if the vector v is non-zero.

Given a pair of vectors $v_1, v_2 \in \mathbb{R}^3$, we may consider the linear subspace given by all linear combinations of them

$$X := \{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

We note that X is a plane unless v_1 and v_2 are parallel. These describe *every* linear subspace of $\mathbb{R}^3$.

More generally, for any linear subspace $X \subseteq \mathbb{R}^n$ which is not just the trivial subspace $\{0\}$, there is a collection of linearly independent vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ for some $1 \leq k \leq n$, such that

$$X = \left\{ \sum_{j=1}^{k} \lambda_j v_j : \lambda_j \in \mathbb{R},\ \forall j \in \{1, \ldots, k\} \right\}.$$

We say that X has dimension k, written $\dim(X) = k$.

Remark. *If $X \subseteq \mathbb{R}^n$ is a linear subspace, then it must contain the origin $0 \in X$. We leave it as an easy exercise for the reader to prove this.*

Consider a straight line $L \subseteq \mathbb{R}^2$ not passing through the origin. By the previous remark, we see that L cannot be a linear subspace, even though geometrically it looks identical to one. However, if we translate L by a fixed vector so that it does contain the origin, then L will be a linear subspace. This leads to the following definition

Definition 1.5. *$X \subseteq \mathbb{R}^n$ is called affine (or affine linear) if there exists some $a \in \mathbb{R}^n$ such that*

$$X + a := \{x + a : x \in X\},$$

is a linear subspace of $\mathbb{R}^n$.

In this section, we find multiple ways of describing such subspaces. The easiest examples to begin with are lines in the plane.

Example 12. Consider the line $L \subseteq \mathbb{R}^2$ passing through the points $(0, 2)$ and $(1, 0)$. The usual way of representing such a line is via the equation:

$$2x + y = 2.$$

We also have the following parametric form of L: given the two points on L, we can view the line as the equation we get by starting at one of the points (e.g $(1, 0)$) and moving some amount $t \in \mathbb{R}$ along the vector formed given by the difference of the second point with the first you are at (e.g $(0, 2) - (1, 0) = (-1, 2)$). Indeed we may describe L as the set of vectors

$$L = \{(1, 0) + t(-1, 2) : t \in \mathbb{R}\}.$$

Note that we can easily recover the equation for the line from the parametric form by eliminating the parameter t,

$$x = 1 - t, y = 2t, \ \forall t \in \mathbb{R} \iff 1 - x = \frac{y}{2} \iff 2x + y = 2.$$

In the parametric form, it is also obvious that the line L is affine,

$$L + (-1, 0) = \{t \cdot (-1, 2) : t \in \mathbb{R}\}.$$

Lines in $\mathbb{R}^n$

Let $L \subseteq \mathbb{R}^n$ be a line. Suppose $A \in L$ is a point on the line and $v \in \mathbb{R}^n$ is the direction of L. That is, $v \in L - \overrightarrow{OA}$. Then the parametric form of the line is given by

$$L = \{\overrightarrow{OA} + tv : t \in \mathbb{R}\},$$

where the real number $t \in \mathbb{R}$ is called a parameter.

Example 13. Suppose $L \subseteq \mathbb{R}^3$ is the line passing through the points $A = (1, 2, 3)$ and $B = (-1, 3, 5)$.

In this case, we may take

$$v = \overrightarrow{AB} = (-1, 3, 5) - (1, 2, 3) = (-2, 1, 2),$$

so that the parametric form of the line is

$$L = \{(1, 2, 3) + t \cdot (-2, 1, 2) : t \in \mathbb{R}\}.$$

To find the equation for the line L, we solve for t:

$$\frac{1 - x}{2} = \frac{y - 2}{1} = \frac{z - 3}{2} \ (= t).$$

Remark. *The parametric form of a line is not unique.*

- *Choosing a different initial point on the line corresponds to translating the value of t by a fixed amount:*
  In the previous example, we have chosen $A = (-1, 3, 5)$, giving the parametric form of the line
  $$L = \{(-1, 3, 5) + s \cdot (-2, 1, 2) : s \in \mathbb{R}\}.$$
  This is the same as the original parameterisation by setting $t = s - 1$.
- *Choosing a stretched version of our direction vector corresponds to sclaing the value of t by a fixed amount:*
  In the previous example, we have chosen $v = (4, -2, -4)$, giving the parametric form of the line
  $$L = \{(1, 2, 3) + s \cdot (4, -2, -4) : s \in \mathbb{R}\}.$$
  This is the same as the original parameterisation by setting $t = -2s$.

Planes in $\mathbb{R}^3$

A plane $P \subseteq \mathbb{R}^3$ is determined by

(i) Three non-colinear points on P;

(ii) A point on P, and two linearly independent directions in P;

(iii) A point on P and a normal vector to P.

Here, v is a direction in P if $v \in P - \overrightarrow{OA}$ for any point $A \in P$, and n is a normal vector to P if $n \perp v$ for any direction v in P.

For case ii), if $A \in P$ and $u, v \in \mathbb{R}^3$ be linearly independent directions in P, then we have the following parametric form of the plane

$$P = \{\overrightarrow{OA} + tu + sv : t, s \in \mathbb{R}\}.$$

For case iii), if $A = (a_1, a_2, a_3) \in P$ and $n = (n_1, n_2, n_3)$ is a normal vector to P, then for any $x \in \mathbb{R}^3$ we see that

$$x \in P \iff x - \overrightarrow{OA} \perp n \iff (x - \overrightarrow{OA}) \cdot n = 0 \iff x \cdot n = \overrightarrow{OA} \cdot n,$$

and hence the equation of the plane is

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = \underbrace{a_1 n_1 + a_2 n_2 + a_3 n_3}_{\in \mathbb{R}},$$

where the right hand side is just a constant.

In particular, for any non-zero vector $(a, b, c) \in \mathbb{R}^3$, the set of points $(x, y, z) \in \mathbb{R}^3$ solving the equation

$$ax + by + cz = d, \quad (d \in \mathbb{R}),$$

is a plane with normal vector (a, b, c). We note that if $(a, b, c) = 0 \in \mathbb{R}^3$, then this equation describes either all of $\mathbb{R}^3$ when $d = 0$, or the empty set when $d \neq 0$.

Example 14. Let P be the plane passing through the points $A = (0, 0, 1)$, $B = (0, 2, 0)$, and $C = (-1, 1, 0)$. To find a parametric form of P, we note that the vectors

$$\overrightarrow{AB} = (0, 2, -1), \quad \overrightarrow{AC} = (-1, 1, -1),$$

are directions in P, and therefore

$$P = \{(0, 0, 1) + s(0, 2, -1) + t(-1, 1, -1) : s, t \in \mathbb{R}\}.$$

Next we note that the vector

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1, 1, 2),$$

is a normal vector to P, and so the equation of the plane is given by

$$-x + y + 2z = (-1, 1, 2) \cdot (x, y, z) = (-1, 1, 2) \cdot (0, 0, 1) = 2.$$

For two subsets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, we define the distance between these subsets to be the infimum of the distance between pairs of points in these sets

$$\inf_{A \in \mathcal{A}, B \in \mathcal{B}} \|\overrightarrow{AB}\|.$$

The following example shows how to calculate the distance between a point and a plane.

Example 15. Let $A = (2, 1, 1)$ and P be the plane given by the equation

$$-x + 2y - z = -4, \tag{1.7}$$

so that P has normal vector $n = (-1, 2 - 1)$. Consider the line

$$L := \{A + tn = (2 - t, 1 + 2t, 1 - t) : t \in \mathbb{R}\},$$

and let $B = L \cap P$ be the intersection point of this line and the plane.

Fact: B is the point on P closest to A (see Homework 2).

To find B, substitute our formula for a point on L into (1.7)

$$(-1)(2 - t) + 2(1 + 2t) - (1 - t) = -4 \iff t = -\frac{1}{2},$$

and hence $B = (\frac{5}{2}, 0, \frac{3}{2})$. Therefore, the distance between A and P is

$$\|\overrightarrow{AB}\| = \|(1/2, -1, 1/2)\| = \sqrt{6}/2.$$

Exercise: Find the distance between the lines

$$L_1 = \{(-4, 9, -4) + s(4, -3, 0) : s \in \mathbb{R}\},$$
$$L_2 = \{(5, 2, 10) + t(4, 3, 2) : t \in \mathbb{R}\}.$$

Hint: Find $A \in L_1$, $B \in L_2$ such that $\overrightarrow{AB} \perp L_1, L_2$.

Note that lines in $\mathbb{R}^3$ may be viewed as the intersection of two planes:

Example 16. Consider the line

$$x + y + 6z = 6, \quad x - y - 2z = -2.$$

Then, performing Gaussian elimination of this system of equations, we find a general solution is

$$x = 2 - 2t, y = 4 - 4t, z = t, \quad t \in \mathbb{R},$$

which is a parametric form of the line. Conversely, we can eliminate the parameter t from the parametric form to find

$$2x - y = 0, \quad y + 4z = 4,$$

which is another set of equations defining the same line.

Example 17. Given three non-trivial planes in $\mathbb{R}^3$, what is the intersection of those planes?

Case 1: A single unique point. Note that in this case, the normal vectors to the three planes are linearly independent.

Case 2: A line or a plane. Note that in this case, the normal vectors to the three planes are linearly dependent.

Case 3: The empty set. Similar to the previous case, the normal vectors to the three planes are linearly dependent.

General Affine Subspaces

Given a non-zero vector $n \in \mathbb{R}^n$, the solutions $x \in \mathbb{R}^n$ to the equation

$$n \cdot x = \lambda, \quad (\lambda \in \mathbb{R})$$

describes a hyperplane (a dimension $n - 1$ affine subspace) with normal vector n.

To describe a dimension k plane P in $\mathbb{R}^n$, we can either describe it using its

- Parametric form:

  $$P = \{q + \sum_{j=1}^{k} t_j v_j : t_1, \ldots, t_k \in \mathbb{R}\},$$

  where $q \in P$, $v_1, \ldots, v_k$ are linearly independent vectors in the direction of P, and $t_1, \ldots, t_k \in \mathbb{R}$ are parameters.

- Equations:

  $$\sum_{j=1}^{n} a_{ij} x_j = c_i, \quad \forall i \in \{1, \ldots, n - k\},$$

  where $c = (c_1, \ldots, c_{n-k}) \in \mathbb{R}^{n-k}$, $A = (a_{ij})$ is a $(n - k) \times n$ matrix with full rank (trivial kernel).

  Note that a system of $(n-k)$ equations is equivalent to the intersection of $(n-k)$ hyperplanes. In this language, the condition of the matrix A is equivalent to the normal vectors to the $(n - k)$ hyperplanes being linearly independent.

Exercise: Show that in the parametric form of P, we may choose the direction vectors $v_1, \ldots, v_k$ to be orthonormal, i.e.

$$v_i \cdot v_j = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}, \quad \forall i, j \in \{1, \ldots, k\}.$$

Hint: If you are struggling, look up the Gram-Schmidt algorithm and apply it here.

1.3 Curves

Definition 1.6. *Let $I \subseteq \mathbb{R}$ be an interval. A curve in $\mathbb{R}^n$ is a continuous function $x : I \to \mathbb{R}^n$. That is,*

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \quad \forall t \in I,$$

where every component function $x_i : I \to \mathbb{R}$, for $i \in \{1, \ldots, n\}$, is continuous.

Example 18. $x : [-1, 1) \to \mathbb{R}^2$, $x(t) = (t, t^2)$. This traces a section of the parabola $y = x^2$ starting at the point $(-1, 1)$ and moving in the positive x direction up to (but not including) the point $(1, 1)$.

Example 19. $x : \mathbb{R} \to \mathbb{R}^3$, $x(t) = p + tq$ for some $p, q \in \mathbb{R}^3$ $(q \neq 0)$. This is a parameterisation of a straight line as we saw earlier.

In the following definition, we restrict our attention to curves whose domain is a closed interval $I = [a, b] \subseteq \mathbb{R}$.

Definition 1.7. *A curve $x : [a, b] \to \mathbb{R}^n$ is said to be*

(a) ***closed*** *if $x(a) = x(b)$.*

(b) ***simple*** *if $x(t_1) \neq x(t_2)$ for any $a \leq t_1 < t_2 \leq b$, except for possibly $t_1 = a, t_2 = b$.*

Theorem 20. *Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a curve in $\mathbb{R}^n$. Then*

- $\lim\limits_{t \to a} x(t) = \left(\lim\limits_{t \to a} x_1(t), \ldots, \lim\limits_{t \to a} x_n(t)\right)$.
- $x'(t) = \lim\limits_{h \to 0} \frac{x(t+h) - x(t)}{h} = (x_1'(t), \ldots, x_n'(t))$, *provided the limit exists.*

Remark. *$x'(a)$ is the tangent vector of x at $t = a$. If a curve $x : I \to \mathbb{R}^n$ has a well defined tangent vector at every point $t \in I$, then we say the curve is differentiable. If moreover, the map $x' : I \to \mathbb{R}^n$ is continuous (so the derivative is itself a curve), then we say that the original curve is a C^1-curve.*

Example 21. The curve $x : \mathbb{R} \to \mathbb{R}^2$, $x(t) = (t, |t|)$ is not differentiable as it has no tangent vector at $t = 0$.

Example 22. The curve $x : \mathbb{R} \to \mathbb{R}^2$, $x(t) = (t^2 \sin t^{-1}, 0)$ for $t \neq 0$, and $x(0) = (0, 0)$ is a differentiable curve, but is not a C^1-curve.

We have the following physical interpretation for derivatives of curves:

Suppose $x(t)$ denotes the displacement of a particle at time t. Then $x'(t)$ will denote the particles velocity, $\|x'(t)\|$ its speed, and $x''(t)$ its acceleration at time t.

Example 23. Suppose $x(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Then

$$x'(t) = (-\sin t, \cos t) \perp x(t), \quad \forall t \in [0, 2\pi],$$
$$x''(t) = (-\cos t, -\sin t) = -x(t), \quad \forall t \in [0, 2\pi],$$

and $\|x'(t)\| \equiv 1$.

Example 24. $x : [1, \infty) \to \mathbb{R}^2$, $x(t) = (t^{-1}, t^{-2})$. Then $\lim_{t\to\infty} x(t) = (0, 0)$.

Properties for Derivatives of Curves

Let $x, y : I \to \mathbb{R}^n$ be differentiable curves, $\lambda \in \mathbb{R}$, and $f : I \to \mathbb{R}$ a differentiable function.

a) $(x + y)'(t) = x'(t) + y'(t)$;

b) $(\lambda x)'(t) = \lambda \cdot x'(t)$;

c) $(f \cdot x)'(t) = f'(t)x(t) + f(t)x'(t)$;

d) $(x \cdot y)'(t) = x'(t) \cdot y(t) + x(t) \cdot y'(t)$;

e) If $n = 3$, $(x \times y)'(t) = x'(t) \times y(t) + x(t) \times y'(t)$.

Arclength of a Curve

Suppose $x : I \to \mathbb{R}^n$ is a C^1-curve.

Definition 1.8. *The arclength of the curve $x(t)$ for $a \le t \le b$ is*

$$S := \int_a^b \|x'(t)\| dt.$$

Referring back to the physical interpretation of a curve and its derivatives, we are saying that the integral of the speed between two times is equal to the distance travelled.

Alternatively, we could view the arc-length as a limit of the following approximations:

Take $a = t_0 < t_1 < \ldots < t_n = b$ for some subdivision of the interval $[a, b]$. Then consider the following sequence of line segments which approximate our curve

$$\overrightarrow{x(t_0)x(t_1)},\ \overrightarrow{x(t_1)x(t_2)},\ \ldots,\ \overrightarrow{x(t_{n-1})x(t_n)}.$$

The arclength of the curve S is then approximated by the sum of the length of these line segments

$$S \approx \sum_{j=1}^{n} \|\overrightarrow{x(t_{j-1})x(t_j)}\| = \sum_{j=1}^{n} \|x(t_j) - x(t_{j-1})\|.$$

Since x is differentiable, we also have the approximation

$$x'(t_j) = \lim_{h\to 0} \frac{x(t_j) - x(t_j + h)}{h} \approx \frac{x(t_j) - x(t_{j-1})}{t_j - t_{j-1}},$$

and therefore

$$S \approx \sum_{j=1}^{n} \|x'(t_j)\|(t_j - t_{j-1}).$$

Taking finer and finer approximations, the sum converges to an integral against dt and we recover the desired formula.

Example 25. Consider the helix parameterised by $x(t) = (\cos t, \sin t, t)$, for $t \in [0, 2\pi]$.

Its tangent vector at each t is $x'(t) = (-\sin t, \cos t, 1)$, and so the tangent line of x when $t = \pi$ has parameterisation

$$x(\pi) + s \cdot x'(\pi) = (-1, 0, \pi) + s \cdot (0, -1, 1), \quad \forall s \in \mathbb{R}.$$

Moreover, the speed of the curve is constant

$$\|x'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Thus, the arclength of the helix is

$$S = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Example 26. Consider the following two curves $x : [0, 4] \to \mathbb{R}^2$, $x(t) = (t, t)$ and $y : [0, 2] \to \mathbb{R}^2$, $y(t) = (t^2, t^2)$. We note that these two curves are two different parameterisations of the line segment joining the origin to the point $(2, 2)$ with their speeds always positive away from the end points. They should therefore give the same arc length value. Indeed

$$\int_0^4 \|x'(t)\|\, dt = \int_0^4 \|(1, 1)\|\, dt = \int_0^4 \sqrt{2}\, dt = 4\sqrt{2},$$

$$\int_0^2 \|y'(t)\|\, dt = \int_0^2 \|(2t, 2t)\|\, dt = \int_0^2 2\sqrt{2}t\, dt = 4\sqrt{2}.$$

Definition 1.9. *A C^1-curve $x : (a, b) \to \mathbb{R}^n$ is called regular if*

$$\|x'(t)\| > 0, \quad \forall t \in (a, b).$$

Exercise: Prove that the following independence of parameterisation for the arc length: If $x : (a, b) \to \mathbb{R}^n$, $y : (c, d) \to \mathbb{R}^n$ are two regular C^1-curves with the same image

$$\{x(t) \in \mathbb{R}^n : t \in (a, b)\} = \{y(s) \in \mathbb{R}^n : s \in (c, d)\},$$

show that the arc length of x is equal to the arc length of y.

Polar Coordinates

Given a point $P = (x, y) \in \mathbb{R}^2$, we can represent it by the pair of values

$$\begin{cases} r = \sqrt{x^2 + y^2} = \text{ the distance from the origin,} \\ \theta = \text{ the anticlockwise angle between the x-axis and } \overrightarrow{OP}. \end{cases}$$

Example 27. The point $P = (1, 1)$ has $r = \sqrt{2}$ and $\theta = \frac{\pi}{4} + 2\pi k$, for $k \in \mathbb{Z}$.

We note that for the origin, $r = 0$, and θ is not uniquely defined.

Remark. *There are different conventions for the ranges of r and θ. In this course we choose the convention $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$, so that (r, θ) are uniquely determined away from the origin. Although we make this convention, it is sometimes useful to allow $r < 0$ as we shall see later.*

Given polar coordinates (r, θ) it is easy to read off the Cartesian coordinates directly

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Conversely, to read off the polar coordinates from the Cartesian coordinates, we have to be careful on the signs of x and y when determining θ. For example, when $x, y > 0$, then

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

However, when $x, y < 0$, then we choose $\theta = \frac{\pi}{2} + \arctan(\frac{y}{x})$ instead, so that θ always lies in the region $[0, 2\pi)$.

Curves in Polar Coordinates

We explore some examples of curves in the plane represented using polar coordinates.

Example 28. The equation $r = r_0$ determines a circle of radius $r_0 > 0$ centred at the origin.

Example 29. The equation $\theta = \theta_0$ determines the half ray starting from (and including) the origin which makes an angle of θ_0 with the positive x-axis in the anticlockwise direction.

Example 30. The Archimedes spiral $r = k\theta$ for any constant $k > 0$.

Example 31. The equation $r = 4 \cos \theta$ determines a circle centred at $(2, 0)$ of radius 2. To see this we can change back to Cartesian coordinates. We first note that the origin satisfies this equation, so when multiplying by r we do not change the equation.

$$r = 4 \cos \theta \iff r^2 = 4r \cos \theta \iff x^2 + y^2 = 4x \iff (x - 2)^2 + y^2 = 2.$$

We note that under our convention that $r \geq 0$, we must restrict $\theta \in [0, \frac{\pi}{2}] \cap [\frac{3\pi}{2}, 2\pi)$.

Example 32. The equation $r\cos(\theta - \frac{\pi}{4}) = \sqrt{2}$ determines a straight line passing through $(0, -2)$ and $(2, 0)$. To see why we again change back to Cartesian coordinates. Expanding the right hand side using the double angle formula gives

$$2 = \sqrt{2}r\left(\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}\right) = r\cos\theta + r\sin\theta = x + y.$$

Recall that our convention is $r \geq 0$, however in some instances, allowing $r < 0$ can be convenient. In particular, for $r < 0$,

$$\begin{aligned}(x, y) &= (r\cos(\theta), r\sin(\theta)) \\ &= (-|r|\cos(\theta), -|r|\sin(\theta)) \\ &= (|r|\cos(\theta + \pi), |r|\sin(\theta + \pi)).\end{aligned}$$

Example 33. With this convention, the curve $\{\theta = \theta_0\}$ is now an entire straight line.

Example 34. Consider the curve $r = 1 - \lambda\cos\theta$ for some $\lambda > 1$.

Case 1: We begin with the convention $r \geq 0$.

$$1 - \lambda\cos\theta \geq 0 \implies \cos\theta \leq \lambda^{-1} < 1 \implies \theta \in [\delta, 2\pi - \delta],$$

for $\delta := \arccos(\lambda^{-1})$. Sketching this curve gives a simple cardioid curve.

Case 2: If we allow $r \in \mathbb{R}$ instead, then we have an extra loop for the region $\theta \in [-\delta, \delta]$. In particular, the curve is now self-intersecting.

Coordinates in $\mathbb{R}^3$

One way to incorporate polar coordinates in higher dimensions is to use them within a plane, and leave the final Cartesian coordinate as it is. These are known as cylindrical coordinates.

To express a point $P = (x, y, z) \in \mathbb{R}^3$ in cylindrical coordinates, we convert the (x, y) part of P into polar coordinates. That is, we choose (r, θ, z) given by the relation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Example 35. The helix can be easily parameterised in cylindrical coordinates:

$$r = 1, \quad \theta = t, \quad z = t, \quad \forall t \in [0, 2\pi).$$

We also have a higher dimensional version of polar coordinates known as spherical coordinates. That is, we describe a point $P = (x, y, z) \in \mathbb{R}^3$ by:

$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} = \text{ the distance from the origin,} \\ \theta = \text{ the anticlockwise angle between the x-axis and } (x, y, 0), \\ \varphi = \text{ the angle between the positive z-axis and } (x, y, z). \end{cases}$$

We note that $\varphi \in [0, \pi]$. To read off the spherical coordinates from the cartesian coordinates, we decompose $P = (x, y, 0) + (0, 0, z)$, and note that $z = \rho \cos \varphi$, and $(x, y, 0)$ has length $\rho \sin \varphi$. Therefore, we find that

$$x = \rho \sin \varphi \cos \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \varphi.$$

Example 36. The subset $\rho = 2$ corresponds to the sphere centred at the origin of radius 2.

Example 37. The subset $\varphi = \frac{\pi}{4}$ corresponds to the cone with cone angle $\frac{\pi}{4}$, its tip at the origin, lying within the upper half space $\{z > 0\}$.

Example 38. The subset $\theta = 0$ corresponds to the half plane $\{x > 0, y = 0\}$.

Week 2

2.1 Topological Terminology

Definition 2.10. *Given a point $x_0 \in \mathbb{R}^n$ and a fixed constant $r > 0$, we define the open ball centred at x_0 of radius r to be*

$$B_r(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\| < r\}.$$

Similarly, we define the closed ball centred at x_0 of radius r to be

$$\overline{B_r(x_0)} := \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}.$$

Definition 2.11. *Let $S \subseteq \mathbb{R}^n$ be any subset.*

- *The interior of S is the set*

$$\text{Int}(S) := \{x \in \mathbb{R}^n : B_r(x) \subseteq S, \text{ for some } r > 0\}.$$

  Points in Int(S) *are called interior points of S.*

- *The exterior of S is the set*

$$\text{Ext}(S) := \{x \in \mathbb{R}^n : B_r(x) \subseteq \mathbb{R}^n \setminus S, \text{ for some } r > 0\}.$$

  Points in Ext(S) *are called exterior points of S.*

- *The exterior of S is the set*

$$\partial S := \{x \in \mathbb{R}^n : B_r(x) \cap S \neq \emptyset \text{ and } B_r(x) \cap (\mathbb{R}^n \setminus S) \neq \emptyset, \ \forall r > 0\}.$$

  Points in ∂S are called boundary points of S.

Example 39. Consider the annulus $S = \left\{(r, \theta) \in \mathbb{R}^2 : 1 < r \leq 4\right\}$, where (r, θ) denotes polar coordinates.

Let A be a point lying on the inner circle $r = 1$, B be a point lying inside the annulus with $r \in (1, 4)$, C be a point on the outer circle $r = 4$, and D a point with $r > 4$.

We note that $A, D \notin S$, $B, C \in S$, and that

- $A \in \partial S$ is a boundary point of S;
- $B \in \text{Int}(S)$ is an interior point of S;
- $C \in \partial S$ is a boundary point of S;

- $D \in \text{Ext}(S)$ is an exterior point of S.

Let $S \subseteq \mathbb{R}^n$. The following properties follow directly from the definition:

1. $\mathbb{R}^n$ is the disjoint union of $\text{Int}(S)$, $\text{Ext}(S)$ and ∂S;
2. $\text{Int}(S) \subseteq S$ and $\text{Ext}(S) \subseteq \mathbb{R}^n \setminus S$. A point in ∂S may or may not be in S.

Definition 2.12. *A subset $S \subseteq \mathbb{R}^n$ is called*

*1. **open** if $\forall x \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.*

*2. **closed** if $\mathbb{R}^n \setminus S$ is open.*

Equivalently, S is open if $S = \text{Int}(S)$, and S is closed if $S = \text{Int}(S) \cup \partial S$.

Remark. *There are precisely two subsets of $\mathbb{R}^n$ which are both open and closed: $\mathbb{R}^n$ and $\emptyset$. Some subsets of $\mathbb{R}^n$ are neither open nor closed, e.g the annulus $\{1 < r \leq 4\}$ from before.*

Note that, for any $S \subseteq \mathbb{R}^n$, then it is always true that $\text{Int}(S)$ and $\text{Ext}(S)$ are both open, and hence ∂S is closed.

Bounded and Path Connected Subsets

Definition 2.13. *$S \subseteq \mathbb{R}^n$ is **bounded** if $\exists M > 0$ such that*

$$S \subseteq B_M(0) = \{x \in \mathbb{R}^n : \|x\| < M\}.$$

S is unbounded if S is not bounded.

Definition 2.14. *S is **path connected** if any two points in S can be connected by a curve in S, i.e $\forall x, y \in S$, there exists a curve $\gamma : [0, 1] \to S$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Remark. *There is also a notion of connectedness which is slightly more subtle that we shall not discuss in these notes. It is a fact that if S is path connected, then S is connected, however the converse is does not hold: there exists sets S which are connected but not path connected.*

Theorem 40 (Jordan curve theorem)**.** *A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two path connected components, one bounded and one unbounded.*

Remark. *Although this seems trivial, the result is surprisingly hard to prove. In fact, an analogous statement in higher dimensions which was believed to be true for many years is actually false. (For the interested reader, you may look up the Jordan–Schönflies theorem and the Alexander horned sphere.)*

2.2 Vector-Valued Multivariable Functions

We now consider functions $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. One way to visualise such a function is by looking at the graph of f:

$$\mathrm{Graph}(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \Omega\}.$$

Alternatively, we may consider the level sets of f: for any $c \in \mathbb{R}^m$, we define the level set at c to be

$$L_c := f^{-1}(c) = \{x \in \Omega : f(x) = c\} \subseteq \Omega \subseteq \mathbb{R}^n.$$

Example 41. For the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x + y$, and for any $c \in \mathbb{R}$, we have that the level set L_c is the line given by the equation $y = c - x$.

Example 42. For the function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x^2 + y^2$, and for any $c \in \mathbb{R}$, we have that

$$L_c = \begin{cases} \emptyset & : c < 0 \\ (0,0) & : c = 0 \\ \text{the circle } \partial B_{\sqrt{c}}(0) & : c > 0. \end{cases}$$

Example 43. For the function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x, y) = \cos(2\pi(x^2 + y^2))$, the level set L_1 is given by those points (x, y) such that $x^2 + y^2$ is an integer. Therefore

$$L_1 = \{0\} \cup \bigcup_{k \geq 1} \partial B_{\sqrt{k}}(0).$$

Limits of Multivariable Functions

Let $A \subseteq \mathbb{R}^n$. We define $\overline{A} = A \cup \partial A$, known as the closure of A. Since $A \subseteq \mathrm{Int}(A) \cup \partial A$, we see that

$$\overline{A} = \mathrm{Int}(A) \cup \partial A = \mathbb{R}^n \setminus \mathrm{Ext}(A),$$

and hence $\overline{A}$ is closed.

Fact: $\overline{A}$ is the smallest closed set containing A (See Homework 3).

For any point $a \in \overline{A}$ and any function $f : A \to \mathbb{R}^m$, we now define a notion of a limit of $f(x)$ as $x \to a$.

Definition 2.15. *For $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $a \in \overline{A}$ and $L \in \mathbb{R}^m$, we say that $\lim_{x \to a} f(x) = L$ if, $\forall \epsilon > 0$, $\exists \delta > 0$ such that,*

$$x \in A \quad \text{with} \quad 0 < \|x - a\| < \delta \quad \implies \quad \|f(x) - L\| < \epsilon.$$

Remark. *This is the same definition of a limit as in the 1-dimensional case, only with the absolute value $|\cdot|$ being replaced by the length of vectors $\|\cdot\|$. Here, you should read the quantity $\|x - y\|$ as the distance between x and y.*

The fact that we require $\|x - a\| > 0$ means we do not care about the value of $f(a)$ if it exists.

Example 44. Take $f(x, y) = x + y$. We will prove using the definition that

$$\lim_{(x,y)\to(2,1)} f(x, y) = 3.$$

That is, we must show that for any $\epsilon > 0$, we can find a $\delta > 0$ (which may depend on ϵ) such that, if

$$0 < \|(x, y) - (2, 1)\| < \delta,$$

then

$$|f(x, y) - 3| = |x + y - 3| < \epsilon.$$

We first notice that the quantity we want to control can be bound above using the triangle inequality:

$$|x + y - 3| \leq |x - 2| + |y - 1| . \tag{2.8}$$

Moreover, we have that each of these terms is controlled by the distance between (x, y) and $(2, 1)$:

$$|x - 2| , |y - 1| \leq \sqrt{(x - 2)^2 + (y - 1)^2} = \|(x, y) - (2, 1)\|. \tag{2.9}$$

Therefore, if we picked $\epsilon = 1$ for example, then choosing $\delta = 1/2$ we see that

$$0 < \|(x, y) - (2, 1)\| < 1/2$$

implies by (2.9) that

$$|x - 2| , |y - 1| < 1/2 \implies |x - 2| + |y - 1| < 1,$$

which by (2.8) that

$$|x + y - 3| < 1,$$

as required.

More generally, if we fix $\epsilon > 0$ to be an arbitrary positive number, we then choose $\delta = \epsilon/2 > 0$. It then follows from (2.8) and (2.9) that if $0 < \|(x, y) - (2, 1)\| < \delta$, then $|x + y - 3| < \epsilon$. Since the initial $\epsilon > 0$ chosen was arbitrarily, we have shown it to be true for any $\epsilon > 0$.

Example 45. Take $f(x, y) = x^2 + y^2$. We will prove using the definition that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$

Fix $\epsilon > 0$ and choose $\delta = \sqrt{\epsilon} > 0$. If $0 < \|(x, y) - (0, 0)\| = \|(x, y)\| < \delta$, we have

$$|f(x, y) - 0| = \left|x^2 + y^2\right| = \|(x, y)\|^2 < \delta^2 = \epsilon.$$

Component Functions

Let $A \subseteq \mathbb{R}^n$, $a \in \overline{A}$, $f : A \to \mathbb{R}^m$. We can decompose the vector $f(x)$ at every point $x \in A$:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix},$$

where each $f_i : A \to \mathbb{R}$ for $i = 1, \ldots, m$ is called a component of f.

Lemma 46. *Let $f : A \to \mathbb{R}^m$, $a \in \overline{A}$ and $L \in \mathbb{R}^m$. Then*

$$\lim_{x \to a} f(x) = L = (\ell_1, \ldots, \ell_m) \in \mathbb{R}^m \quad \iff \quad \lim_{x \to a} f_j(x) = \ell_j, \quad \forall j = 1, \ldots, m.$$

As a consequence of this lemma, it is enough to focus on limits of $\mathbb{R}$-valued functions ($m = 1$).

Proof. We begin with the ($\Longrightarrow$) direction. Fix $\epsilon > 0$. By the definition of $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that $x \in A$ with $0 < \|x - a\| < \delta$ implies that

$$\left|f_j(x) - \ell_j\right| \leq \sqrt{\sum_{i=1}^{k} (f_i(x) - \ell_i)^2} = \|f(x) - L\| < \epsilon,$$

for each $j = 1, \ldots, m$. This is precisely what we wanted to show.

For the reverse direction ($\Longleftarrow$), we again fix $\epsilon > 0$. Consider the new value $\tilde{\epsilon} = \frac{\epsilon}{\sqrt{m}} > 0$. For each j, plugging this value of $\tilde{\epsilon}$ into the definition of $\lim_{x \to a} f_j(x) = \ell_j$, we have that there exists $\delta_j > 0$ such that, if $x \in A$ and if $0 < \|x - a\| < \delta_j$, then $\left|f_j(x) - \ell_j\right| < \tilde{\epsilon}$.

We now set $\delta := \min\{\delta_1, \ldots, \delta_m\} > 0$. Then, if $x \in A$ and if $0 < \|x - a\| < \delta$, we have that

$$\|f(x) - L\| = \sqrt{\sum_{j=1}^{k} \left|f_j(x) - \ell_j\right|^2} \leq \sqrt{\sum_{j=1}^{k} \tilde{\epsilon}^2} = \sqrt{m}\tilde{\epsilon} = \epsilon.$$

This is precisely the definition of $\lim_{x \to a} f(x) = L$.

□

Paths in the Domain

When the domain is a single variable, the ways we may approach at point $a \in \mathbb{R}$ are limited. In particular, there are only two directions to consider; from above and from below. Therefore

$$\lim_{x \to a} f(x) \text{ exists} \iff \underbrace{\lim_{x \uparrow a} f(x) = \lim_{x \downarrow a} f(x)}_{\text{(they exists and are equal)}}.$$

When our domain is in $\mathbb{R}^n$ for $n \geq 2$, there are now infinitely many ways to approach a point $a \in \mathbb{R}^n$. When defining a limit, we need to consider *every* possible curve along which we can approach a. In particular, $\lim_{x \to a} f(x) = L$ if and only if, the limit of $f(x)$ when x approaches a along any path exists and equals L.

Therefore, in order to show that such a limit does *not* exist, we can either:

- Find a path along which the limit does not exists;
- Find two paths such that the limit along these paths is different.

In both cases, we may conclude that the limit does not exist (DNE).

Example 47. Consider the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$. In order to investigate the limit of f at the origin, we look at the limits of f along different paths to the origin.

(i) Along the x-axis, we have

$$\lim_{(x,0)\to(0,0)} \frac{x^2 - 0}{x^2 + 0} = \lim_{x\to 0} 1 = 1.$$

(ii) Along the y-axis, we have

$$\lim_{(0,y)\to(0,0)} \frac{0 - y^2}{0 + y^2} = \lim_{y\to 0} -1 = -1.$$

Since $1 \neq -1$, we conclude that the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ DNE.

Example 48. Consider the function $g : \mathbb{R}^2 \setminus \{(1, 2)\} \to \mathbb{R}$, $g(x, y) = \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$. To investigate the limit of g at $(1, 2)$, we approach $(1, 2)$ along the line with slope $m \in \mathbb{R}$. i.e along

$$y = 2 + m(x - 1).$$

We first note that

$$g(x, y) = \frac{xy - 2x - y + 2}{(x - 1)^2 + (y - 2)^2} = \frac{(x - 1)(y - 2)}{(x - 1)^2 + (y - 2)^2},$$

and so

$$g(x, m(x - 1) + 2) = \frac{m(x - 1)^2}{(x - 1)^2 + m^2(x - 1)^2}.$$

Therefore, taking a limit along this line gives

$$\begin{aligned}\lim_{(x,m(x-1)+2)\to(1,2)} g(x, m(x - 1) + 2) &= \lim_{x\to 1} \frac{m(x - 1)^2}{(x - 1)^2 + m^2(x - 1)^2}\\ &= \lim_{x\to 1} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}.\end{aligned}$$

Since this value is different for different choices of $m \in \mathbb{R}$, the limit $\lim_{(x,y)\to(1,2)} g(x, y)$ DNE.

Example 49. $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x, y) = \begin{cases} 1 & : 0 < y < x^2 \\ 0 & : \text{else.} \end{cases}$$

Investigate the limit of f at the points

$$a = (0, 1), \quad b = (1, 1), \quad c = (0, 0).$$

- Near the point a, $f \equiv 0$, and so

$$\lim_{(x,y)\to a} f(x,y) = 0.$$

- We note that

$$f(x,1) = \begin{cases} 1 & : x > 1 \\ 0 & : x \le 1. \end{cases}$$

Therefore,

$$\lim_{x\downarrow 1} f(x,1) = 0, \quad \lim_{x\uparrow 1} f(x,1) = 1,$$

and since these are different the limit of f at b DNE.

- Along any line $y = mx$ with $m > 0$ we have that

$$f(x,mx) = \begin{cases} 1 & : x > m \\ 0 & : x \le m. \end{cases}$$

Therefore, $\lim_{(x,mx)\to(0,0)} f(x,mx) = 0$, for all $m > 0$. A similar calculation shows the same for $m \le 0$.

However, this does **not** mean that the limit at c exists and is zero. To see why, consider instead the curve $y = \frac{1}{2}x^2$. At every point on this curve (except the origin) $f \equiv 1$. Therefore,

$$\lim_{(x,\frac{x^2}{2})\to(0,0)} f(x, \frac{x^2}{2}) = 1 \neq 0,$$

and so the limit of f at c DNE.

Properties of Limits

Lemma 50. *For each of the following equations, if the limits on the right hand side (RHS) exist, then the limit on the left hand side (LHS) exists, and the equation holds.*

(i)

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

(ii)

$$\lim_{x\to a} \lambda f(x) = \lambda \lim_{x\to a} f(x), \quad (\text{for } \lambda \in \mathbb{R}).$$

(iii)

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

(iv)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \text{if } \lim_{x\to a} g(x) \neq 0.$$

(v)

$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n, \quad \forall n \in \mathbb{N}_0.$$

(vi)

$$\lim_{x\to a}(f(x))^{\frac{1}{n}} = \left(\lim_{x\to a} f(x)\right)^{\frac{1}{n}}, \quad \text{if } n \text{ is even and } f(x) \geq 0 \text{ near a.}$$

Theorem 51 (Squeeze theorem)**.** *Let $f, g, h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. If*

$$g(x) \leq f(x) \leq h(x),$$

for $x \in \Omega$ near $a \in \overline{\Omega}$, and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$

for some $L \in \mathbb{R}$, then $\lim_{x\to a} f(x) = L$.

Remark. *We say that a statement $P(x)$ is true for $x \in \Omega$ near $a \in \overline{\Omega}$ if: $\exists \delta > 0$ such that, if $x \in \Omega$ with $x \in B_\delta(a) \setminus \{a\}$, then $P(x)$ is true.*

Proof of the Squeeze theorem. Fix $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that

$$0 < \|x - a\| < \delta_1, x \in \Omega \implies h(x) - L < \epsilon.$$

Similarly, there exists $\delta_2 > 0$ such that

$$0 < \|x - a\| < \delta_2, x \in \Omega \implies L - g(x) < \epsilon.$$

Let $\delta_3 > 0$ be sufficiently small so that

$$0 < \|x - a\| < \delta_3, x \in \Omega \implies g(x) \leq f(x) \leq h(x).$$

Setting $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$, we see that, if $0 < \|x - a\| < \delta$ and $x \in \Omega$, then

$$\begin{cases} f(x) - L \leq h(x) - L < \epsilon, \\ L - f(x) \leq L - g(x) < \epsilon, \end{cases} \implies |f(x) - L| < \epsilon. \qquad \square$$

We have the following Corollary to the Squeeze theorem

Corollary 1. *Let $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. If*

$$|f(x)| \leq g(x),$$

for $x \in \Omega$ near $a \in \overline{\Omega}$, and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} f(x) = 0$.

Example 52. Consider the function $f(x, y) = x \cos\left(\frac{1}{x^2+y^2}\right)$. Since

$$|f(x, y)| = |x| \underbrace{\left|\cos\left(\frac{1}{x^2 + y^2}\right)\right|}_{\leq 1} \leq |x|,$$

and $\lim_{(x,y)\to(0,0)} |x| = 0$, by the Squeeze theorem, $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

Example 53. Consider the function $g(x,y) = \frac{(x-1)^2 \log x}{(x-1)^2+y^2}$. Since

$$|g(x,y)| = |\log x| \underbrace{\left|\frac{(x-1)^2}{(x-1)^2+y^2}\right|}_{\leq 1} \leq |\log x|,$$

and $\lim_{(x,y)\to(1,0)} |\log x| = |\log 1| = 0$, by the Squeeze theorem, $\lim_{(x,y)\to(1,0)} g(x,y) = 0$.

Finding Limits in Polar Coordinates

Note that in polar coordinates, $(x,y) \to (0,0)$ is equivalent to $r \downarrow 0$.

Example 54.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\downarrow 0} \frac{r^3(\sin^3\theta + \cos^3\theta)}{r^2} = \lim_{r\downarrow 0} r(\sin^3\theta + \cos^3\theta) = 0,$$

where the final equality comes from the bound

$$\left|\sin^3\theta + \cos^3\theta\right| \leq \left|\sin^3\theta\right| + \left|\cos^3\theta\right| \leq 2,$$

and the Squeeze theorem.

Example 55.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)} = \lim_{r\downarrow 0} \frac{r^2\cos^2\theta + r^2\cos\theta\sin\theta}{2r^2} == \lim_{r\downarrow 0} \frac{\cos^2\theta + \cos\theta\sin\theta}{2},$$

which depends on θ. Therefore, the limit DNE.

Example 56.

$$\lim_{(x,y)\to(0,0)} xy\log(x^2+y^2) = \lim_{r\downarrow 0} 2r^2\cos\theta\sin\theta\log r.$$

Since $\left|2r^2\cos\theta\sin\theta\log r\right| \leq \left|2r^2\log r\right|$, and

$$\begin{aligned}\lim_{r\downarrow 0} 2r^2\log r &= \lim_{r\downarrow 0} \frac{2\log r}{r^{-2}} \quad (= \frac{-\infty}{\infty})\\ &= \lim_{r\downarrow 0} \frac{2r^{-1}}{-2r^{-3}}\\ &= \lim_{r\downarrow 0} -r^2 = 0,\end{aligned}$$

where in the second equality, we have use L'Hôpital's rule. Applying the Squeeze theorem, we conclde the limit is zero.

Iterated Limits

Question: when are the following limits equal?

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right), \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right), \quad \lim_{(x,y)\to(0,0)} f(x,y).$$

Example 57. If $f(x,y) = \frac{x+y}{x-y}$, then

$$\begin{aligned}
&\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\frac{x}{x} = 1;\\
&\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}\frac{y}{-y} = -1;\\
&\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{DNE.}
\end{aligned}$$

Example 58. If

$$f(x,y) = \begin{cases} 1 & : x = y,\\ 0 & : x \neq y,\end{cases}$$

then we have that

$$\begin{aligned}
&\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0;\\
&\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0;\\
&\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{DNE.}
\end{aligned}$$

In particular, this example shows that just because the iterated limits exist and are equal:

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right),$$

does not imply that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

Example 59. If

$$f(x,y) = \begin{cases} x\cos y^{-1} + y\cos x^{-1} & : x, y \neq 0,\\ 0 & : \text{else,}\end{cases}$$

then

$$\begin{aligned}
&\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) \text{DNE;}\\
&\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) \text{DNE;}\\
&\lim_{(x,y)\to(0,0)} f(x,y) = 0.
\end{aligned}$$

This example shows that if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, this does not imply that the iterated limits

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right), \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right),$$

must exist. However, it is true that if all three limits exists, then they must coincide.

2.3 Continuity

Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in A$. We give two different definitions for the function f being continuous at the point a. The first is an $\epsilon - \delta$ definition, whereas the second uses the notion of a limit.

Definition 2.16. *f is continuous at a if*

- $\forall \epsilon > 0$, $\exists \delta > 0$ *such that, if $x \in A$ and $\|x - a\| < \delta$, then $|f(x) - f(a)| < \epsilon$.*
- $\lim_{x\to a} f(x)$ *exists and equals $f(a)$.*

f is continuous on A if f is continuous at a, for every $a \in A$.

Example 60. Fix $k \in \{1, \dots, n\}$, and let $F_k : \mathbb{R}^n \to \mathbb{R}$ denote the coordinate function

$$F_k(x) = F_k((x_1, \dots, x_n)) = x_k.$$

Fix $a \in \mathbb{R}^n$ and $\epsilon > 0$. We then choose $\delta = \epsilon > 0$. If $\|x - a\| < \delta$, then

$$|F_k(x) = F_k(a)| = |x_k - a_k| \le \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} = \|x - a\| < \delta = \epsilon.$$

Therefore, F_k is continuous at $a \in \mathbb{R}^n$, and hence F_k is continuous on $\mathbb{R}^n$ for every $k = 1, \dots, n$.

The following theorem follows directly from the properties of limits.

Theorem 61. *If $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ are continuous functions at $a \in \Omega$, then*

1. *$f + g$, λf, fg are continuous at a ($\lambda \in \mathbb{R}$).*
2. *$\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.*

Combining this theorem with the previous example of the coordinate function being continuous, we see that all polynomials and rational functions are continuous

Example 62. The function $f(x, y, z) = x^3 + 3yz + z^2 - x + 7y$ is continuous on $\mathbb{R}^3$, and the function $g(x, y, z) = \frac{x^3+y^3+yz}{x^2+y^2}$ is continuous on $\mathbb{R}^3 \setminus \{x = y = 0\}$.

Given a rational function $Q(x) = \frac{P_1(x)}{P_2(x)}$ for some polynomials P_1, P_2, we see that Q is continuous away from the zero level set of P_2. That is, $Q : \mathbb{R}^n \setminus \{P_2 = 0\} \to \mathbb{R}$ is continuous.

Suppose $P_2(a) = 0$. Then Q can be extended to a continuous function at a if and only if the limit $\lim_{x\to a} Q(x)$ exists.

Example 63. If $Q(x, y) = \frac{xy+y^3}{x^2+y^2}$, then Q is continuous on $\mathbb{R}^2 \setminus \{0\}$. Fix $m \ge 0$ and consider

$$\lim_{(x,mx)\to(0,0)} Q(x, mx) = \lim_{x\to 0} \frac{mx^2 + m^3x^3}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{m + m^3x}{1 + m^2} = \frac{m}{1 + m^2}.$$

Since this varies in m, the limit $\lim_{(x,y)\to(0,0)} Q(x, y)$ DNE, and hence Q cannot be extended to a continuous function on all of $\mathbb{R}^2$.

Example 64. If $Q(x,y) = \frac{x^4-y^4-5x^2y^2}{x^2+y^2}$, then Q is continuous on $\mathbb{R}^2 \setminus \{0\}$. We note that

$$\lim_{(x,y)\to(0,0)} Q(x,y) = \lim_{r\downarrow 0} \frac{r^4(\cos^4\theta - \sin^4\theta - 5\sin^2\theta\cos^2\theta)}{r^2} = \lim_{r\downarrow 0} r^2 \underbrace{(\cos^4\theta - \sin^4\theta - 5\sin^2\theta\cos^2\theta)}_{\text{bounded in absolute value by 7}} = 0,$$

where in the final equality we have used the Squeeze theorem. Therefore Q can be extended to a continuous function on all of $\mathbb{R}^2$, by setting

$$Q(x,y) := \begin{cases} \frac{x^4-y^4-5x^2y^2}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : x = y = 0. \end{cases}.$$

Theorem 65. *If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous at $a \in \Omega$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(a) \in \mathbb{R}$, then $g \circ f : \Omega \to \mathbb{R}$ is continuous at a, and hence*

$$\lim_{x\to a} g \circ f(x) = g\left(\lim_{x\to a} f(x)\right) = g \circ f(a).$$

Proof. Fix $\epsilon > 0$. Since g is continuous, there exists $\eta > 0$ such that for $y \in \mathbb{R}$, if $|y - f(a)| < \eta$, then $|g(y) - g \circ f(a)| < \epsilon$. Then, since f is continuous, there exists $\delta > 0$ such that, for $x \in \Omega$, if $\|x - a\| < \delta$, then $|f(x) = f(a)| < \eta$, which then implies that $|g \circ f(x) - g \circ f(a)| < \epsilon$ as required. □

Example 66. Letting $f = F_k$, the k-coordinate function from before, and $g(x) = |x|$, we see that the maps

$$(x_1, \ldots, x_n) \mapsto |x_k| \text{ are continuous, for every } k = 1, \ldots, n.$$

Example 67. Choosing the same f as the previous example, but $g(x) = |\log x|$, which is continuous on $(0, \infty)$, we see that for any $k \in \{1, \ldots, n\}$, the map

$$(x_1, \ldots, x_n) \mapsto |\log x_k| \text{ is continuous on the half-space } \{x_k > 0\}.$$

Example 68.

$$\sin(x^2 + yz), \quad e^{x-y}, \quad r = \sqrt{x^2 + y^2},$$

are all continuous functions everywhere in their domains.

2.4 Partial Derivatives

We now consider the rate of change of a function with respect to each variable individually.

Definition 2.17. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ with Ω open. For $i \in \{1, \ldots, n\}$, we define the i^{th}-partial derivative of f at $x \in \Omega$ to be*

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h},$$

where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n$ is the unit vector with all components equal to zero except for the i^{th}-component, which is 1.

Example 69. For the function $f(x, y) = x^2 + y^2$, we have

$$\frac{\partial f}{\partial x} = 2x \quad \text{(regard } y \text{ as a constant)}$$
$$\frac{\partial f}{\partial y} = 2y \quad \text{(regard } x \text{ as a constant)}$$

We note that, at the point $(1, -1)$, we have $\frac{\partial f}{\partial x}(1, -1) = 2 > 0$, $\frac{\partial f}{\partial y}(1, -1) = -2 < 0$, so

$$f \text{ is increasing as } x \text{ increases at } (1, -1),$$
$$f \text{ is decreasing as } y \text{ increases at } (1, -1).$$

Example 70. For the function $g(x, y, z) = xy^2 - \cos(xz)$, we have

$$\frac{\partial g}{\partial x} = g_x = y^2 + z\sin(xz)$$
$$\frac{\partial g}{\partial y} = g_y = 2xy$$
$$\frac{\partial g}{\partial z} = g_z = x\sin(xz)$$

Example 71. In this example we consider the function

$$f(x, y) = \begin{cases} 1 & : xy \geq 0 \\ 0 & : xy < 0 \end{cases}.$$

To calculate the partial derivative f_x, we fix $y \in \mathbb{R}$ and differentiate with respect to x:

- Fix $y = 1$. Then
$$f(x, 1) = \begin{cases} 1 & : x \geq 0 \\ 0 & : x < 0 \end{cases},$$
and hence $f_x(1, 1) = 0$, $f_x(0, 1)$ DNE.
- Fix $y = 0$. Then $f(x, 0) \equiv 1$ and hence $f_x(x, 0) \equiv 0$.

Similarly, $f_y(0, 0) = 0$. However, f is **not** continuous at the origin.

The previous example shows that just because the partial derivatives exist at a point does not imply that the function is continuous at this point.

Higher Order Derivatives

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, there are exactly two first order partial derivatives f_x, f_y. However, we may then take partial derivatives of these functions and find four different second order partial derivatives of f:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial(f_x)}{\partial x},$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial(f_x)}{\partial y} = f_{xy},$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(f_y)}{\partial x} = f_{yx},$$
$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial(f_y)}{\partial y}.$$

Repeating, we find that f has 2^k different partial derivatives of order k. e.g. f_{yyx} is a partial derivative of order three.

Example 72. Find all first and second order partial derivatives of the function $f(x, y) = x \sin y + y^2 e^{2x}$:

$$f_x = \sin y + 2y^2 e^{2x}$$
$$f_{xx} = 4y^2 e^{2x}$$
$$f_{xy} = \cos y + 4ye^{2x}$$
$$f_y = x \cos y + 2ye^{2x}$$
$$f_{yx} = \cos y + 4ye^{2x}$$
$$f_{yy} = -x \sin y + 2e^{2x}$$

In the previous example, $f_{xy} = f_{yx}$. It turns out however, that this is not always true.

Example 73. Let

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & : (x, y) \neq (0, 0) \\ 0 & : x = y = 0 \end{cases}.$$

For $y \neq 0$, $f(x, y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ near $(0, y)$, and therefore near $(0, y)$ we have

$$f_x(x, y) = \frac{(3x^2 y - y^3)(x^2 + y^2) - 2x^2 y(x^2 - y^2)}{(x^2 + y^2)^2}.$$

Substituting $x = 0$ into the above equation, we conclude that

$$f_x(0, y) = \frac{-y^5}{y^4} = -y, \quad \forall y \neq 0.$$

Alternatively, we have that

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and hence $f_x(0,y) = -y$ for every $y \in \mathbb{R}$, from which we deduce that $f_{xy}(0,0) = -1$.

Alternatively, f has asymmetry about the line $y = x$. That is

$$f(b,a) = -f(a,b), \quad \forall (a,b) \in \mathbb{R}^2.$$

In particular,

$$f_y(b,a) = \lim_{h\to 0} \frac{f(b,a+h) - f(b,a)}{h} = \lim_{h\to 0} -\frac{f(a+h,b) - f(a,b)}{h} = -f_x(a,b),$$

and similarly, $f_{yx}(b,a) = -f_{xy}(a,b)$. Substituting in $a = b = 0$, we find that $f_{yx}(0,0) = -f_{xy}(0,0) = -1$.

This example shows $f_{xy} \neq f_{yx}$ in general. The following theorem gives a sufficient condition on f for these mixed partial derivatives to agree.

Theorem 74 (Clairaut's Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^2$ open, and $f : \Omega \to \mathbb{R}$. If both the partial derivatives f_{xy} and f_{yx} exist and are continuous everywhere in Ω, then $f_{yx} = f_{xy}$ on Ω.*

Remark. *To prove the theorem, we actually prove a slightly stronger version that what is stated in the theorem: $\Omega \subseteq \mathbb{R}^2$ open, $f : \Omega \to \mathbb{R}$, $a \in \Omega$. If both f_{xy} and f_{yx} exist in an open ball containing a, and are continuous at a, then $f_{xy}(a) = f_{yx}(a)$.*

In order to prove Clairaut's theorem, we shall use the Mean Value Theorem, the proof of which can be found in any introductory mathematical analysis course.

Theorem 75 (Mean Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists some $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof of Clairaut's Theorem. We may assume that $a = (0,0) \in \Omega$. Let $h,k > 0$ so that $[0,h] \times [0,k] \subseteq \Omega$. We consider the constant

$$\alpha = f(h,k) - f(0,k) - f(h,0) + f(0,0).$$

We now apply the Mean Value Theorem (MVT) in both the x and y directions separately. That is, consider the function $g : [0,h] \to \mathbb{R}$, given by $g(x) = f(x,k) - f(x,0)$. By the MVT, we have that there exists some $h_1 \in (0,h)$ such that

$$g'(h_1) = \frac{g(h) - g(0)}{h}.$$

Note that the LHS $g'(h_1) = f_x(h_1, k) - f_x(h_1, 0)$, and the denominator of the RHS $g(h) - g(0) = f(h,k) - f(h,0) - f(0,k) + f(0,0) = \alpha$. Therefore, we have

$$f_x(h_1, k) - f_x(h_1, 0) = \alpha h^{-1}.$$

Next, we consider the function $G : [0, k] \to \mathbb{R}$, given by $G(y) := f_x(h_1, y)$. Applying MVT again, $\exists k_1 \in (0, k)$ such that

$$f_{xy}(h_1, k_1) = G_y(k_1) = \frac{G(k) - G(0)}{k} = \frac{f_x(h_1, k) - f_x(h_1, 0)}{k} = \alpha(hk)^{-1}.$$

We now repeat the process but we swap the order of derivatives. That is, there exists $(h_2, k_2) \in (0, h) \times (0, k)$ such that

$$f_{yx}(h_2, k_2) = \alpha(hk)^{-1},$$

and therefore $f_{yx}(h_2, k_2) = f_{xy}(h_1, k_1)$. If we then take $h, k \downarrow 0$, this forces $(h_1, k_1), (h_2, k_2) \to (0, 0)$. We can then use the continuity of f_{xy} and f_{yx} to conclude that $f_{yx}(0, 0) = f_{xy}(0, 0)$. $\square$

Definition 2.18. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}$, $r \in \mathbb{N}_0$. We call f a C^r-function if all partial derivatives of f up to order r exist and are continuous on Ω.*

f is called a smooth function, or a C^∞-function if it is a C^r-function for every $r \geq 0$.

Example 76. f is a C^0-function if it is a continuous function.

Example 77. $f(x, y)$ is a C^2-function if all of the functions $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exist and are continuous.

Example 78. The following class of functions are smooth within their domain of existence: Polynomials, Rational functions, Exponentials, Logarithms, and Trignometric functions.

Example 79. Sums, differences, products, quotients and compositions of all of the classes of functions from the previous example are also smooth within their domain of existence. e.g. $\exp(x^2 - y)\sin(\frac{x}{y})$ is a smooth function on $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$.

The following Corollary of Clairaut's theorem holds by repeated application of the theorem to a function and its derivatives.

Corollary 2. *If $f : \Omega \to \mathbb{R}$ is a C^r-function on the open set $\Omega \subseteq \mathbb{R}^n$ for some $r \geq 0$. Then the order of taking partial derivatives does not matter for all partial derivatives up to order r.*

Example 80. If $f(x, y, z)$ is a C^3-function, then

$$f_{xz} = f_{zx}, \quad f_{xyz} = f_{zxy} = f_{yzx}, \quad f_{xxy} = f_{xyx} = f_{yxx}.$$

Week 3

3.1 Differentiability

In one dimension, $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if the following limit exists

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In the multivariable, or higher dimensional domain situation $f : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$, why can't we use the same defintion?

$$\lim_{x \to a} \frac{\overbrace{f(x) - f(a)}^{\in \mathbb{R}}}{\underbrace{x - a}_{\in \mathbb{R}^n}}.$$

We cannot divide a real number by a vector in $\mathbb{R}^n$! In order to formulate a suitable definition in this situation, we reconsider the single variable case again.

Affine Approximations

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$. Then, for x close to a, we have that

$$f(x) \approx L(x) := f(a) + f'(a)(x - a),$$

where $L(x)$ is the *best* affine function (polynomial of degree one or less) to approximate $f(x)$ about the point a.

To make this notion of best approximation more precise, let us consider the error function:

$$\begin{aligned} \varepsilon(x) &:= f(x) - L(x) \\ &= f(x) - f(a) - f'(a)(x - a). \end{aligned}$$

Since $x - a \in \mathbb{R}$, we can divide through by it to see that

$$\frac{\varepsilon(x)}{x - a} = \frac{f(x) - f(a)}{x - a} - f'(a),$$

and so by definition, taking $x \to a$ we have

$$\lim_{x \to a} \frac{\varepsilon(x)}{x - a} = f'(a) - f'(a) = 0,$$

or equivalently

$$\lim_{x \to a} \frac{\varepsilon(x)}{|x-a|} = 0. \tag{3.10}$$

That is, the error function is small compared to the distance between x and a.

Exercise: Show that this choice of affine approximation is the only one for which (3.10) holds.

In higher dimensions, the graph of f should be approximated by a higher dimensional affine object (e.g the tangent plane of $z = f(x, y)$).

Example 81. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and assume $f_x(a, b)$ and $f_y(a, b)$ exist. Then we could take the affine approximation at (a, b) of f to be:

$$f(x, y) \approx L(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

In particular, $z = L(x, y)$ is a hyperplane touching Graph(f) at the point $(a, b, f(a, b))$.

This leads to the following definition

Definition 3.19. *Let $\Omega \subseteq \mathbb{R}^n$ open, $a \in \Omega$, $f : \Omega \to \mathbb{R}$. Then, f is said to be differentiable at a if*

- *All partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exist, for $i \in \{1, \ldots, n\}$.*
- *Given the affine approximation of f at a*

$$f(x) = \underbrace{f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)}_{L(x)} + \underbrace{\varepsilon(x)}_{\text{error}},$$

the error term satisfies $\lim_{x \to a} \frac{\varepsilon(x)}{\|x-a\|} = 0$.

That is, f is differentiable at a if f can be well-approximated by an affine function locally about the point a.

Remark.

- *$L(x)$ is a degree one or less polynomial in the variables $x_1, \ldots, x_n$.*
- *$L(a) = f(a)$ and $\frac{\partial L}{\partial x_i}(a) = \frac{\partial f}{\partial x_i}(a)$.*
- *$y = L(x)$ is an n-dimensional hyperplane in $\mathbb{R}^{n+1}$ tangent to Graph(f) at $(x, f(x))$.*

Example 82. Let $f(x, y) = x^2 y$.

(i) Show that f is differentiable at $(1, 2)$.

(ii) Approximate $f(1.1., 1.9)$ using the derivative.

(iii) Find the tangent plane of $z = f(x, y)$ at the point $(1, 2, f(1, 2))$.

To show (i), we calculate the first order partial derivatives of f:

$$f_x = 2xy, \quad f_y = x^2,$$

and hence, at the point $(1,2)$ we have $f_x(1,2) = 4$, $f_y(1,2) = 1$. Our affine approximation at $(1,2)$ is then

$$\begin{aligned} L(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 2 + 4(x-1) + (y-2). \end{aligned}$$

Let $\varepsilon(x,y)$ denote the corresponding error function. Since

$$\begin{aligned} \lim_{(x,y)\to(1,2)} \frac{\varepsilon(x,y)}{\|(x,y)-(1,2)\|} &= \lim_{(x,y)\to(1,2)} \frac{x^2y - 2 - 4(x-1) - (y-2)}{\|(x,y)-(1,2)\|} \qquad (h = x-1, k = y-2) \\ &= \lim_{(h,k)\to(0,0)} \frac{(1-h)^2(2+k) - 2 - 4h - k}{\|(h,k)\|} \\ &= \lim_{(h,k)\to(0,0)} \frac{h^2k + 2hk + 2h^2}{\sqrt{h^2+k^2}} \\ &= \lim_{r\downarrow 0} \frac{r^3\cos^2\theta\sin\theta + 2r^2\cos\theta\sin\theta + 2r^2\cos^2\theta}{r} \\ &= \lim_{r\downarrow 0} r^2\cos^2\theta\sin\theta + 2r\cos\theta\sin\theta + 2r\cos^2\theta = 0, \end{aligned}$$

where the last equality is due to the Squeeze theorem. Therefore, f is differentiable at $(1,2)$.

For (ii), $f(1.1, 1.9) \approx L(1.1, 1.9) = 2 + 4(0.1) + 1(-0.1) = 2.3$.

Finally, the tangent plane for part (iii) is given by $z = L(x,y)$. In particular, $4x + y - z = -4$, so that the tangent plane has normal vector $(4, 1, -1)$.

Example 83. Is the function $f(x,y) = \sqrt{|xy|}$ differentiable at the origin?

Note that $f(x,0) = 0$ for every $x \in \mathbb{R}$, and $f(0,y) = 0$ for every $y \in \mathbb{R}$. So $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$, and similarly $f_y(0,0) = 0$.

Therefore, if f can be approximated by an affine function at the origin, the approximation is $L(x,y) \equiv 0$, with error function $\varepsilon = f$. However,

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|} = \lim_{r\downarrow 0} \frac{\sqrt{r^2\,|\cos\theta\sin\theta|}}{r} = \lim_{r\downarrow 0} \sqrt{|\cos\theta\sin\theta|},$$

which DNE, and therefore f is **not** differentiable at the origin.

Remark. *In the previous example, along the line $y = mx$, we see that*

$$f(x, mx) = \sqrt{|m|}\,|x|\,.$$

Therefore, along the x-axis $(m = 0)$, we have

$$f(x,0) = 0 = L(x,0),$$

and L is a good approximation to f. However, along the line $y = x$ ($m = 1$) we have

$$f(x, x) = |x| \neq 0 = L(x, x),$$

and L is a bad approximation to f.

In the definition of differentiability, $L(x)$ is defined using only the derivative along the coordinate axes $\frac{\partial f}{\partial x_i}$. We therefore conclude that a function f is differentiable if the information in the coordinate directions can tell you information in every direction (compare this with the previous example).

Unlike the existence of partial derivatives by itself, a function being differentiable is strong enough to imply continuity of the function at that point.

Theorem 84. *If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.*

Proof. Let $L(x)$ denote the affine function approximation of $f(x)$ at a, and $\varepsilon = f - L$. Since f is differentiable at a,

$$\lim_{x \to a} \varepsilon(x) = \lim_{x \to a} \frac{\varepsilon(x)}{\|x - a\|} \cdot \lim_{x \to a} \|x - a\| = 0 \cdot 0 = 0.$$

Therefore,

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} L(x) + \lim_{x \to a} \varepsilon(x) \\ &= f(a) + \lim_{x \to a} \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) \right) + 0 \\ &= f(a), \end{aligned}$$

and f is continuous at a. □

Exercise: Show that the affine function $L : \mathbb{R}^n \to \mathbb{R}$ defined by

$$L(x) = \lambda + x \cdot \alpha, \quad \forall x \in \mathbb{R}^n,$$

for some $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$ is differentiable everywhere, directly from the definition.

Rules for Diffentiation

Theorem 85. *If $\Omega \subseteq \mathbb{R}^n$ open, and $f, g : \Omega \to R$ are differentiable at $a \in \Omega$, then*

(i) $f + g$, λf, $f \cdot g$ are differentiable at a.

(ii) $\frac{f}{g}$ is differentiable at a provided $g(a) \neq 0$.

The proof of this theorem is very similar to those of one-variable (e.g see Math2050).

Example 86. Constant functions are differentiable.
Coordinate functions, e.g $x \mapsto x_k$, are differentiable.
Polynomials, e.g $4x^3y + xy^2 - xyz + z^5$, are differentiable.
Rational functions, e.g $\frac{x^3y+z}{x^2+y^2+z^2+1}$, are differentiable.

The following theorem provides a simpler way to verify differentiability for more regular functions.

Theorem 87. *Let $\Omega \subseteq \mathbb{R}^n$ be open with f a C^1-function on Ω. Then f is differentiable on Ω.*

Remark. *We require all of the partial derivatives to exist and be continuous on the entire open set Ω, and not just at a single point.*

Example 88. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and that f_x, f_y exist and are continuous on a small open ball $B_\epsilon(0)$ about the origin. Then f is differentiable on $B_\epsilon(0)$. In particular, f is differentiable at the origin $0 \in \mathbb{R}^2$.

Example 89. Let $f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = xe^{x+y} - \log(x+z)$, where $\Omega = \{x + z > 0\}$ open. Since

$$\begin{aligned} f_x &= (1+x)e^{x+y} - (x+z)^{-1} \\ f_y &= xe^{x+y} \\ f_z &= -(x+z)^{-1}, \end{aligned}$$

are all continuous functions, f is C^1 on Ω, and hence f is differentiable on Ω.

Proof. We give the proof for $n = 2$. The proof for $n \geq 3$ is exactly the same but with messier notation.

Suppose $(a,b) \in \Omega \subseteq R^2$ and $B_\delta((a,b)) \subseteq \Omega$ for some $\delta > 0$. For any point $(x,y) \in B_\delta((a,b))$, we can apply the MVT to find some k lying between b and y, and some h lying between x and a, so that

$$\begin{aligned} f(x,y) - f(a,b) &= (f(x,y) - f(x,b)) + (f(x,b) - f(a,b)) \\ &= \quad f_y(x,k)(y-b) \quad + \quad f_x(h,b)(x-a). \end{aligned}$$

Therefore, we can use the partial derivatives at these points to bound the error function in the following way

$$\begin{aligned} \frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} &= \frac{\left|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)\right|}{\|(x,y)-(a,b)\|} \\ &= \frac{\left|(f_y(x,k) - f_y(a,b))(y-b) + (f_x(h,b) - f_x(a,b))(x-a)\right|}{\|(x,y)-(a,b)\|} \\ &\leq \left|f_y(x,k) - f_y(a,b)\right| \cdot \frac{|y-b|}{\|(x,y)-(a,b)\|} + |f_x(h,b) - f_x(a,b)| \cdot \frac{|x-a|}{\|(x,y)-(a,b)\|} \\ &\leq \left|f_y(x,k) - f_y(a,b)\right| + |f_x(h,b) - f_x(a,b)|. \end{aligned}$$

Taking $(x, y) \to (a, b)$ will force $(x, k), (h, b) \to (a, b)$, and so by the continuity of f_x and f_y and the Squeeze theorem,

$$\lim_{(x,y)\to(a,b)} \frac{|\varepsilon(x, y)|}{\|(x, y) - (a, b)\|} = 0.$$

This is indeed the definition of f being differentiable at (a, b). $\square$

3.2 Gradient and Directional Derivatives

Definition 3.20. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $a \in \Omega$, $f : \Omega \to \mathbb{R}$. We define the gradient vector of f at a to be*

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right) \in \mathbb{R}^n.$$

Example 90. If $f(x, y) = x^2 + 2xy$, then its gradient is given by

$$\nabla f(x, y) = (f_x, f_y) = (2x + 2y, 2x),$$

and so at the point $(1, 2) \in \mathbb{R}^2$, its gradient is the vector $\nabla f(1, 2) = (6, 2) \in \mathbb{R}^2$.

If a function f is differentiable at a point a, then its affine approximation at that point can be expressed more succinctly

$$\begin{aligned} L(x) &= f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_i - a_i) \\ &= f(a) + \nabla f(a) \cdot (x - a). \end{aligned}$$

Definition 3.21. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $a \in \Omega$, $f : \Omega \to \mathbb{R}$. For any unit vector $u \in \mathbb{R}^n$ ($\|u\| = 1$), the directional derivative of f in the direction u at a is*

$$D_u f(a) = \lim_{h\to 0} \frac{f(a + hu) - f(a)}{h},$$

i.e, the rate of change of f in direction u at the point a.

Example 91. Choosing $u = e_i \in \mathbb{R}^n$, we recover the partial derivative $D_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a)$, for any $i = 1, \dots, n$.

Lemma 92. *If f is differentiable at a, and $u \in \mathbb{R}^n$ is a unit vector, then*

$$D_u f(a) = \nabla f(a) \cdot u.$$

Proof. Since f is differentiable at a, we have

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \varepsilon(x), \tag{3.11}$$

for some error function $\varepsilon(x)$ satisfying $\lim_{x\to a} \frac{\varepsilon(x)}{\|x-a\|} = 0$. Setting $x = a + hu$ in (3.11), we find that

$$f(a + hu) - f(a) = h\,(\nabla f(a) \cdot u) + \varepsilon(a + hu),$$

and therefore

$$\begin{aligned} D_u f(a) &= \lim_{h\to 0}\frac{f(a+hu)-f(a)}{h} \\ &= \nabla f(a)\cdot u + \lim_{h\to 0}\frac{\varepsilon(a+hu)}{h} = \nabla f(a)\cdot u. \end{aligned} \qquad \square$$

For a non-zero vector $v \in \mathbb{R}^n$, then the direction of v is the unit vector $\frac{v}{\|v\|}$.

Example 93. Let $f(x,y) = \arcsin(\frac{x}{y})$. Find the rate of change of f at the point $(1,\sqrt{2})$ in the direction of $v = (1,-1)$.

Solution: Set $u = \frac{v}{\|v\|} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ to be the direction of v. We note that

$$f_x(x,y) = \frac{1}{\sqrt{y^2-x^2}}, \quad f_y(x,y) = \frac{-x}{y\sqrt{y^2-x^2}},$$

and so f, f_x, f_y are all continuous near the point $(1,\sqrt{2})$. In particular f is C^1 near $(1,\sqrt{2})$ and hence f is differentiable at $(1,\sqrt{2})$.

We can therefore use the gradient of f to calculate the desired directional derivative

$$\begin{aligned} D_u f((1,\sqrt{2})) &= \nabla f((1,\sqrt{2}))\cdot u \\ &= (f_x(1,\sqrt{2}), f_y(1,\sqrt{2}))\cdot(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \\ &= \left(1, -\frac{1}{\sqrt{2}}\right)\cdot(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} + \frac{1}{2}. \end{aligned}$$

Geometric Meaning of the Gradient

If f is differentiable at a, $u \in \mathbb{R}^n$ a unit vector, then

$$D_u f(a) = \nabla f(a)\cdot u.$$

By the Cauchy-Schwarz inequality,

$$|\nabla f(a)\cdot u| \le \|\nabla f(a)\|\cdot \underbrace{\|u\|}_{=1} = \|\nabla f(a)\|.$$

Moreover, if $\nabla f(a) \neq 0$, then

$$-\|\nabla f(a)\| \le \nabla f(a)\cdot u \le \|\nabla f(a)\|,$$

where equality holds in the first inequality iff $\nabla f(a) = -\lambda u$ for some $\lambda > 0$, and equality holds in the second inequality iff $\nabla f(a) = \lambda u$ for some $\lambda > 0$.

Indeed, at a, $f(x)$ increases (decreases) most rapidly in the direction of $\nabla f(a)$ $(-\nabla f(a))$ at a rate of $\|\nabla f(a)\|$.

Remark. *When we defined directional derivatives, we specified that u was a unit vector. We could have instead defined a directional derivative for a general vector $v \in \mathbb{R}^n$ in the exact same way*

$$D_v f(a) := \lim_{h \to 0} \frac{f(a+hv) - f(a)}{h}.$$

We note that

$$D_v f(a) = \begin{cases} \|v\| D_{\frac{v}{\|v\|}} f(a) & : v \neq 0 \\ 0 & : v = 0 \end{cases},$$

and therefore, we lose no information by restricting our definition of directional derivatives to the case where v is a unit vector.

Total Derivatives

Given a function $f : \Omega \subseteq \mathbb{R}^n \to R$, with Ω open, differentiable at $a \in \Omega$, we consider the affine approximation of $f(x)$ at a,

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \varepsilon(x).$$

If we denote the change in f by $\Delta f = f(x) - f(a)$, and the change in each coordinate by $\Delta x_i = x_i - a_i$, we find that

$$\Delta f \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \Delta x_i.$$

From the fact f is differentiable at a, we see that this approximation holds up to first order:

$$\Delta f - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \Delta x_i = o(\|x - a\|).$$

This first order approximation is denoted by

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) dx^i, \tag{3.12}$$

and is called the **total derivative** of f at a.

Remark. *Technically, one can make sense of (3.12) by interpreting df and dx^i as functions from Ω into the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}$.*

Example 94. Let $V(r, h) = \pi r^2 h$ denote the volume of a cylinder with height h and radius r. Since V is C^1, it is differentiable, with total derivative

$$\begin{aligned} dV &= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\ &= (2\pi r h) dr + (\pi r^2) dh. \end{aligned}$$

Therefore the change in volume of the cylinder when (r, h) goes from $(3, 12)$ to $(3.08, 11.7)$ is roughly given by dV when $dr = 0.08$ and $dh = -0.3$. That is, the change in volume is approximately (up to first order)

$$(2\pi)(3)(12)(0.08) + (\pi)(9)(-0.3) = 3.06\pi \approx 9.61.$$

Summary

Suppose $f : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$.

Types of Derivatives:

- Directional: $D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$, with $\|u\| = 1$.
- Partial: $\frac{\partial f}{\partial x_i}(a) = D_{e_i} f(a)$, for the standard basis $e_i \in \mathbb{R}^n$.
- Gradient: $\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right) \in \mathbb{R}^n$.
- Total: $df(a) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) dx^i$.
- Higher order: e.g $\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = f_{x_i x_j}(a) = f_{ij}(a)$.

f is a C^k-function if f and all of its partial derivatives up to (and including) order k exist and are continuous.

Affine Approximation:

- $L(x) = f(x) + \nabla f(a) \cdot (x - a)$.
- $\varepsilon(x) = f(x) - L(x)$
- $f(x)$ is differentiable at a if $\lim_{x\to a} \varepsilon(x)(\|x - a\|)^{-1} = 0$.

Relations Between Derivatives:

$$C^\infty \supsetneq \cdots \supsetneq C^{k+1} \supsetneq C^k \supsetneq \cdots \supsetneq C^1 \supsetneq C^0.$$

$$\begin{tikzcd}
 & f \text{ is } C^1 \text{ on an open set containing } a \arrow[d, Rightarrow] & \\
 & f \text{ is differentiable at } a \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
D_u f(a) \text{ exists } \forall u \in \mathbb{R}^n \arrow[d, Rightarrow] & & f \text{ is continuous at } a \\
\frac{\partial f}{\partial x_i}(a) \text{ exists } \forall i = 1, \ldots, n & &
\end{tikzcd}$$

Exercise: Find functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

- f is continuous on $\mathbb{R}^n$, but at some point $a \in \mathbb{R}^n$, none of the partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exist.

- At some point $a \in \mathbb{R}^n$, $D_u f(a)$ exists for all unit vectors u, but f is not continuous at a.
- At some point $a \in \mathbb{R}^n$, all of the partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exist, but not every directional derivative $D_u f(a)$ exists.

Week 4

4.1 Jacobian Matrices

We begin with a brief recap on matrix multiplication. Let A be an $k \times m$ matrix, and B a $m \times n$ matrix. Then we can multiply the two matrices to get the $k \times n$ matrix AB. More explicitly, let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{pmatrix}$$

which we view as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^k$. That is, given a vector $x \in \mathbb{R}^m$, we have

$$Ax = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ \vdots \\ a_{k1}x_1 + \cdots + a_{km}x_m \end{pmatrix} \in \mathbb{R}^k. \tag{4.13}$$

Choosing the row vectors

$$\alpha_i = (a_{i1}, \ldots, a_{im}) \in \mathbb{R}^m, \quad \forall 1 \le i \le k,$$

we can rewrite (4.13) as

$$Ax = \begin{pmatrix} -\alpha_1- \\ \vdots \\ -\alpha_k- \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} \alpha_1 \cdot x \\ \vdots \\ \alpha_k \cdot x \end{pmatrix} \in \mathbb{R}^k.$$

Similarly, given a row vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we can multiply B on the left by y to get a vector in $\mathbb{R}^n$.

$$yB = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} y_1 b_{11} + \cdots + y_m b_{m1}, \cdots, y_1 b_{1n} + \cdots + y_m b_{mn} \end{pmatrix} \in \mathbb{R}^n. \tag{4.14}$$

Choosing now the column vectors

$$\beta_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{pmatrix} \in \mathbb{R}^m, \quad \forall 1 \le j \le n,$$

we can rewrite (4.14) as

$$yB = \begin{pmatrix} - y - \end{pmatrix} \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix} = \begin{pmatrix} y \cdot \beta_1, \cdots, y \cdot \beta_n \end{pmatrix} \in \mathbb{R}^n.$$

Therefore, we can write the matrix product AB as

$$AB = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} -\alpha_1- \\ \vdots \\ -\alpha_k- \end{pmatrix} \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \alpha_1 \cdot \beta_1 & \cdots & \alpha_1 \cdot \beta_n \\ \vdots & \ddots & \vdots \\ \alpha_k \cdot \beta_1 & \cdots & \alpha_k \cdot \beta_n \end{pmatrix}$$

Example 95.

$$\begin{aligned} \overbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}^{B} &= \begin{pmatrix} (1,2)\cdot(5,8) & (1,2)\cdot(6,9) & (1,2)\cdot(7,10) \\ (3,4)\cdot(5,8) & (3,4)\cdot(6,9) & (3,4)\cdot(7,10) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}}_{AB} \end{aligned}$$

Given a multivariable function $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we know that we can reduce its complexitiy by studying its component functions

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \in \mathbb{R}^m,$$

with $f_i : \Omega \longrightarrow \mathbb{R}$, for $1 \leq i \leq m$. We now suppose that $\frac{\partial f_i}{\partial x_j}(a)$ exist for each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, for some $a \in \Omega$.

For any fixed $1 \leq i \leq m$, we note that

$$f_i(x) = f_i(a) + \underbrace{\nabla f_i(a) \cdot (x - a)}_{1\times n \quad n\times 1} + \varepsilon_i(x),$$

where we regard the gradient as a row vector and $(x - a)$ as a column vector in order to use matrix multiplication. In particular, we can gather these m equations together in the form

$$\begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(a) \\ \vdots \\ f_m(a) \end{pmatrix} + \underbrace{\begin{pmatrix} -\nabla f_1(a)- \\ \vdots \\ -\nabla f_m(a)- \end{pmatrix}}_{m\times n} \underbrace{\begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix}}_{n\times 1} + \begin{pmatrix} \varepsilon_1(x) \\ \vdots \\ \varepsilon_m(x) \end{pmatrix} \in \mathbb{R}^m.$$

Definition 4.22. *We define the Jacobian matrix of f at a to be the $m \times n$ matrix*

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} = \begin{pmatrix} -\nabla f_1(a)- \\ \vdots \\ -\nabla f_m(a)- \end{pmatrix},$$

and the best affine approximation of f at a to be $L : \mathbb{R}^n \to \mathbb{R}^m$, given by

$$L(x) = f(a) + Df(a)(x - a) \in \mathbb{R}^m, \quad \forall x \in \mathbb{R}^n.$$

Moreover, we say that f is differentiable at a if the error term $\varepsilon(x) = f(x) - L(x)$ satisfies

$$\lim_{x \to a} \frac{\|\varepsilon(x)\|}{\|x - a\|} = 0.$$

Remark. *Note that if f is real valued ($m = 1$), then $Df(a) = \nabla f(a)$. Also, from our discussion of limits of vector-valued functions*

$$\lim_{x \to a} \frac{\|\varepsilon(x)\|}{\|x - a\|} = 0 \iff \lim_{x \to a} \frac{\varepsilon_i(x)}{\|x - a\|}, \quad \forall i \in \{1, \dots, m\}.$$

So f is differentiable at a if and only if all of its components f_i are differentiable at a.

In the language of total derivatives, we have

$$\underbrace{f(x) - f(a)}_{\Delta f} \approx Df(a) \cdot \underbrace{(x - a)}_{\Delta x}$$

That is, we consider the Jacobian as a linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$, mapping $\Delta x \in \mathbb{R}^n$ to $\Delta f \in \mathbb{R}^m$. In particular,

$$df = Df(a)\, dx.$$

Example 96. Let $f(x, y) = ((y + 1)\log x, x^2 - \sin y + 1)$.

a) Find $Df(1, 0)$.

b) Approximate $f(0.9, 0.1)$.

Solution:

a) Calculating partial derivatives we have

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{y+1}{x} & \log(x) \\ 2x & -\cos y \end{pmatrix}.$$

Therefore

$$Df(1, 0) = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

b)

$$\begin{aligned} L(x,y) &= f(1,0) + Df(1,0)\begin{pmatrix} x-1 \\ y-0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x-1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} x-1 \\ 2x-y \end{pmatrix}. \end{aligned}$$

Therefore

$$f(0.9, 0.1) \approx L(0.9, 0.1) = \begin{pmatrix} -0.1 \\ 1.7 \end{pmatrix}.$$

4.2 Chain Rule

Recall, for two differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ of one variable, the chain rule states that

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x),$$

or, alternatively, if we label $y = f(x)$ and $z = g(y)$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

Example 97. If $f(x) = x^2$ and $g(x) = 2x + 1$, then $g \circ f(x) = 2x^2 + 1$. In particular,

$$(g \circ f)'(x) = 4x = 2 \cdot 2x = g'(f(x)) \cdot f'(x).$$

For multivariable functions, a similar expression holds between the Jacobian matrices, where instead of a product we have matrix multiplication.

Theorem 98 (Chain Rule)**.** *Let $f : \Omega_1 \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $g : \Omega_2 \subseteq \mathbb{R}^m \to \mathbb{R}^k$, with Ω_1, Ω_2 open. Suppose that f is differentiable at $a \in \Omega_1$ and g is differentiable at $b = f(a) \in \Omega_2$. Then $g \circ f$ is differentiable at a with*

$$\underbrace{D(g \circ f)(a)}_{k\times n} = \underbrace{Dg(f(a))}_{k\times m} \cdot \underbrace{Df(a)}_{m\times n}.$$

Example 99. Let $f : \mathbb{R} \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(\theta) = (\cos\theta, \sin\theta), \quad g(u,v) = (2uv, u^2 - v^2).$$

Note that $g \circ f : \mathbb{R} \to \mathbb{R}^2$, so that $D(g \circ f)(\theta)$ is a 2×1 matrix for any value of $\theta \in \mathbb{R}$. There are two potential ways to calculate the derivative of this composition

Method 1: Find the composition explicitly.

$$\begin{aligned} g \circ f(\theta) &= g(\cos\theta, \sin\theta) \\ &= (2\cos\theta\sin\theta, \cos^2\theta - \sin^2\theta) \\ &= (\sin 2\theta, \cos 2\theta). \end{aligned}$$

Therefore

$$D(g\circ f)(\theta) = \begin{pmatrix} 2\cos 2\theta \\ -2\sin 2\theta \end{pmatrix}.$$

Method 2: Use the Chain Rule.

$$\begin{aligned} Df(\theta) &= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \\ Dg(u,v) &= \begin{pmatrix} 2v & 2u \\ 2u & -2v \end{pmatrix}, \\ Dg(f(\theta)) &= \begin{pmatrix} 2\sin\theta & 2\cos\theta \\ 2\cos\theta & -2\sin\theta \end{pmatrix}. \end{aligned}$$

By the Chain Rule

$$\begin{aligned} D(g\circ f)(\theta) &= Dg(f(\theta)) \circ Df(\theta) \\ &= \begin{pmatrix} 2\sin\theta & 2\cos\theta \\ 2\cos\theta & -2\sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -2\sin^2\theta + 2\cos^2\theta \\ -4\cos\theta\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} 2\cos 2\theta \\ -2\sin 2\theta \end{pmatrix}. \end{aligned}$$

Example 100. If $f(x,y) = (x^2, 3xy, x+y^2)$ and $g(u,v,w) = \frac{uw}{v}$, then

$$\begin{aligned} Df(x,y) &= \begin{pmatrix} 2x & 0 \\ 3y & 3x \\ 1 & 2y \end{pmatrix}, \\ Dg(u,v,w) &= \left(\frac{w}{v}, \frac{-uw}{v^2}, \frac{u}{v}\right), \end{aligned}$$

Therefore,

$$Df(1,1) = \begin{pmatrix} 2 & 0 \\ 3 & 3 \\ 1 & 2 \end{pmatrix}, \quad Dg(f(1,1)) = Dg(1,3,2) = \left(\tfrac{2}{3}, \tfrac{-2}{9}, \tfrac{1}{3}\right),$$

and

$$D(g\circ f)(1,1) = Dg(f(1,1)) \cdot Df(1,1) = \left(\tfrac{2}{3}, \tfrac{-2}{9}, \tfrac{1}{3}\right) \begin{pmatrix} 2 & 0 \\ 3 & 3 \\ 1 & 2 \end{pmatrix} = (1,0). \tag{4.15}$$

In the previous example, we could regard f as a change of variables from x, y to u, v, w, and hence we could view the function $g \circ f$ as expressing g as a function of x and y. In particular, by a slight abuse of notation, we could read off (4.15) as

$$\frac{\partial g}{\partial x}(1,1) = 1, \quad \frac{\partial g}{\partial y}(1,1) = 0.$$

That is, in classical notation, we can express the chain rule as follows:

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial g}{\partial w} \cdot \frac{\partial w}{\partial x}$$
$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial g}{\partial w} \cdot \frac{\partial w}{\partial x}$$

Example 101. Let $w(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, and

$$x = 3e^t \sin s, \quad y = 3e^t \cos s, \quad z = 4e^t.$$

Find $\frac{\partial w}{\partial s}$ at $s = t = 0$.

Solution: By the chain rule in classical notation, we have

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} \\ &= \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot 3e^t \cos s - \frac{y}{\sqrt{x^2 + y^2 + z^2}} \cdot 3e^t \sin s. \end{aligned}$$

As $s = t = 0$, $(x, y, z) = (0, 3, 4)$ and hence

$$\frac{\partial w}{\partial s}\Big|_{(s,t)=(0,0)} = 0.$$

Example 102. Roy is walking with position at time t given by

$$x(t) = t^3 + 1, \quad y(t) = 2t^2.$$

His altitude is $H(x, y) = x^2 - y^2 + 100$.

a) Is Roy going uphill or downhill at $t = 1$?

b) Which direction should he go instead at time $t = 1$ to move downhill the most quickly?

Solution:

a) We must find $\frac{\partial H}{\partial t}|_{t=1}$. By the Chain rule

$$\begin{aligned} \frac{\partial H}{\partial t} &= \frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial H}{\partial y} \cdot \frac{\partial y}{\partial t} \\ &= (2x)(3t^2) + (-2y)(4t) \\ &= 2(t^3 + 1)(3t^2) - 4t^2(4t) \\ &= 6t^5 - 16t^3 + 6t^2. \end{aligned}$$

Therefore
$$\frac{\partial H}{\partial t}|_{t=1} = 6 - 16 + 6 = -4 < 0,$$
and Roy is going downhill at $t = 1$.

b) At $t = 1$, $(x, y) = (2, 2)$, and $\nabla H(x, y) = (2x, -2y)$. So at time $t = 1$, the gradient of H is given by $\nabla H(2, 2) = (4, -4)$. Therefore H decreases most rapidly in the direction of $-\nabla H(2, 2)$, which is given by the vector
$$u = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$
That is, Roy should travel in the direction u, which if $(1, 0)$ points east and $(0, 1)$ points north, is northwest.

Note that in the previous example
$$\frac{\partial H}{\partial t} = \nabla H \cdot (\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}),$$
which in words can be stated as the rate of change of the altitude is given by the gradient of the altitude dotted with the velocity you are travelling.

Operator Norm

Before proving the chain rule, we make a brief detour regarding the operator norm. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and for a non-zero vector $x \in \mathbb{R}^n$, consider the quantity
$$\frac{||Ax||}{||x||} \in \mathbb{R},$$
which is the length of the vector $Ax \in \mathbb{R}^m$ divided by the length of the vector $x \in \mathbb{R}^n$. Note that since A is linear,
$$A\left(\frac{x}{||x||}\right) = \frac{A(x)}{||x||},$$
and so
$$||A\left(\frac{x}{||x||}\right)|| = \frac{||Ax||}{||x||}.$$
Therefore, we may assume that x is a unit vector and consider the quantity $||A(x)|| \in \mathbb{R}$ instead.

For any unit vector $x \in \mathbb{R}^n$, we note that the i^{th}-component of Ax (for some $1 \leq i \leq m$) satisfies the inequality
$$|(Ax)_i| = |\alpha_i \cdot x| \leq ||\alpha_i|| \cdot \underbrace{||x||}_{=1} = ||\alpha_i||,$$
where α_i denotes the i^{th}-row of A as before. In particular, we have
$$||Ax||^2 = \sum_{i=1}^{m} (Ax)_i^2 \leq \sum_{i=1}^{m} ||\alpha_i||^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = ||A|| < \infty,$$

where the right hand side is the length of A when considered as a vector in the vector space $\mathbb{R}^{nm}$. Hence we have a uniform upper bound on our quantity for any unit vector $x \in \mathbb{R}^n$. This leads to the following definition.

Definition 4.23. *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The operator norm of A is defined to be*

$$\|A\|_{op} := \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.$$

Remark. *Directly from the definition we find that the inequality*

$$\|Ax\| \le \|A\|_{op}\|x\|,$$

*holds for **every** $x \in \mathbb{R}^n$.*

Example 103. Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ Note that for any unit vector v, we have

$$Av = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix},$$

and so

$$\|Av\| = \sqrt{4x^2 + y^2} \le \sqrt{4x^2 + 4y^2} = 2,$$

or $\|A\|_{op} \le 2$. However, since $\|A(e_1)\| = 2$, we can conclude that $\|A\|_{op} = 2$.

Exercise: Suppose A is a $n \times n$ matrix with real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. What is the operator norm $\|A\|_{op}$ in terms of the eigenvalues?

Proof of Chain Rule. Since f is differentiable at a

$$f(x) - f(a) = Df(a)(x-a) + \varepsilon_f(x), \quad \forall x \in \Omega_1, \tag{4.16}$$

with $\lim_{x\to a} \frac{\|\varepsilon_f(x)\|}{\|x-a\|} = 0$. Since g is differentiable at b

$$g(y) - g(b) = Dg(b)(y-b) + \varepsilon_g(y), \quad \forall y \in \Omega_2, \tag{4.17}$$

with $\lim_{y\to b} \frac{\|\varepsilon_g(y)\|}{\|y-b\|} = 0$. Setting $y = f(x)$, $b = f(a)$, and substituting (4.16) into (4.17)

$$\begin{aligned} g\circ f(x) - g\circ f(a) &= Dg(f(a))\big(Df(a)(x-a) + \varepsilon_f(x)\big) + \varepsilon_g(f(x)) \\ &= \underbrace{Dg(f(a))\cdot Df(a)}_{D(g\circ f)(a)}(x-a) + \underbrace{Dg(f(a))\varepsilon_f(x) + \varepsilon_g(f(x))}_{\varepsilon_{g\circ f}(x)}. \end{aligned}$$

Therefore, to finish the proof, it suffices to show that

$$\lim_{x\to a} \frac{\|\varepsilon_{f\circ g}(x)\|}{\|x-a\|} = 0.$$

In order to do this, by the triangle inequality

$$\frac{\|\varepsilon_{f\circ g}(x)\|}{\|x-a\|} \leq \frac{\|Dg(f(a))\varepsilon_f(x)\|}{\|x-a\|} + \frac{\|\varepsilon_g(f(x))\|}{\|x-a\|},$$

so if the limit as $x \longrightarrow a$ of both terms on the right hand side is zero, then we are done by the Squeeze theorem. We first note that

$$\|Dg(f(a)) \cdot \varepsilon_f(x)\| \leq \|Dg(f(a))\|_{op} \cdot \|\varepsilon_f(x)\|,$$

and since f is differentiable at a,

$$\lim_{x\to a} \frac{\|Dg(f(a))\|_{op}\|\varepsilon_f(x)\|}{\|x-a\|} = \|Dg(f(a))\|_{op} \cdot \lim_{x\to a} \frac{\|\varepsilon_f(x)\|}{\|x-a\|} = 0.$$

Therefore by the Squeeze theorem

$$\lim_{x\to a} \frac{\|Dg(f(a)) \cdot \varepsilon_f(x)\|}{\|x-a\|} = 0.$$

Next, we note that if $f(x) = f(a) = b$, then since $\varepsilon_g(b) = 0$, the quantity we want to control $\frac{\|\varepsilon_g(f(x))\|}{\|x-a\|}$ is identically zero, and there is nothing to show. Thus, without loss of generality, we may assume that $f(x) \neq f(a)$ and hence rewrite the quantity as

$$\frac{\|\varepsilon_g(f(x))\|}{\|x-a\|} = \frac{\|\varepsilon_g(f(x))\|}{\|f(x)-f(a)\|} \cdot \frac{\|f(x)-f(a)\|}{\|x-a\|}.$$

Since for x near a, we can bound the quantity

$$\begin{aligned}\frac{\|f(x)-f(a)\|}{\|x-a\|} &= \frac{\|Df(a)(x-a)+\varepsilon_f(x)\|}{\|x-a\|} \\ &\leq \frac{\|Df(a)(x-a)\|}{\|x-a\|} + \frac{\|\varepsilon_f(x)\|}{\|x-a\|} \\ &\leq \|Df(a)\|_{op} + 1 < \infty,\end{aligned}$$

and as

$$\lim_{x\to a} \frac{\|\varepsilon_g(f(x))\|}{\|f(x)-f(a)\|} = \lim_{y\to b} \frac{\|\varepsilon_g(y)\|}{\|y-b\|} = 0,$$

it follows that

$$\lim_{x\to a} \frac{\|\varepsilon_g(f(x))\|}{\|x-a\|} = 0. \quad \square$$

Summary

(i) $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ (one variable, real-valued).

$$Df(x) = \frac{df}{dx} \quad (1\times 1 \text{ matrix})$$

(ii) $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ (multivariable, real-valued).

$$Df(x) = \nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)) \quad (1 \times n \text{ matrix})$$

(iii) $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ (multivariable, vector-valued).

$$Df(x) = \begin{pmatrix} -\nabla f_1(x)- \\ \vdots \\ -\nabla f_m(x)- \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j} \quad (m \times n \text{ matrix})$$

Chain Rule:

$$(x_1, \ldots, x_n) \xrightarrow{f} (y_1, \ldots, y_m) \xrightarrow{g} (g_1, \ldots, g_k),$$

$g_i = g_i(y_1, \ldots, y_m)$ is a function of the variables $y_1, \ldots, y_m$,
$y_j = f_j(x_1, \ldots, x_n)$ is a change of variables from the x_i to the y_j.

Then, the components of the composition $g \circ f$ can be considered as $g_i = g_i(x_1, \ldots, x_n)$; functions of the variables $x_1, \ldots, x_n$. With this set-up, the Chain Rule becomes

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1} & \cdots & \frac{\partial g_k}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial y_1} & \cdots & \frac{\partial g_k}{\partial y_m} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

Looking at the i^{th} row and j^{th} column of the matrix on the left hand side, we find that

$$\frac{\partial g_i}{\partial x_j} = \left(\frac{\partial g_i}{\partial y_1}, \ldots, \frac{\partial g_i}{\partial y_m}\right) \cdot \begin{pmatrix} \frac{\partial y_1}{\partial x_j} \\ \vdots \\ \frac{\partial y_m}{\partial x_j} \end{pmatrix} = \sum_{a=1}^{m} \frac{\partial g_i}{\partial y_a} \cdot \frac{\partial y_a}{\partial x_j}.$$

Applications of the Chain Rule

Theorem 104. $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, *$\Omega$ open, $c \in \mathbb{R}$, $S = f^{-1}(c)$ and $a \in S$. If f is differentiable at a with $\nabla f(a) \neq 0$, then $\nabla f(a) \perp S$ at a.*

Remark. *By $\nabla f(a) \perp S$ at a, we mean that for any tangent vector $v \in \mathbb{R}^n$ to S at a, $\nabla f(a) \perp v$.*

Example 105. If $f(x, y) = x^2 + y^2$, $S = f^{-1}(25)$ is the circle centred at the origin of radius 5, and $a = (4, 3) \in S$, then $\nabla f(x, y) = (2x, 2y)$ and so $\nabla f(4, 3) = (8, 6)$, which is certainly normal to the circle at the point $(4, 3)$.

Example 106. Let $S = \{x^2 + 4y^2 + 9z^2 = 22\}$ be an ellipsoid centred at the origin. In order to find the tangent plane of S at the point $(3, 1, 1)$, we consider the function $f(x, y, z) = x^2 + 4y^2 + 9z^2$ so that $S = f^{-1}(22)$. Then $\nabla f(x, y, z) = (2x, 8y, 18z)$, or $\nabla f(3, 1, 1) = (6, 8, 18)$ which is perpendicular to S at $(3,1,1)$. That is, $(6,8,18)$ is a normal vector to the tangent plane, and hence the tangent plane has equation

$$((x,y,z)-(3,1,1))\cdot(6,8,18)=0,$$

which simplifies to $3x+4y+9z=22$.

Proof. Let $\gamma:(\epsilon,\epsilon)\to S$ be *any* curve inside the level set S such that $\gamma(0)=a$. Then its derivative $\gamma'(0)$ is a tangent vector to S at a (in fact, this is how we technically define tangent vectors to a manifold). Since the curve γ always lies within the level set S, we find that

$$f\circ\gamma(t)=c,\quad \forall t\in(-\epsilon,\epsilon).$$

In particular, $(f\circ\gamma)'(t)=0$ for any $t\in(-\epsilon,\epsilon)$. Alternatively, we can apply the chain rule to find that

$$(f\circ\gamma)'(t)=\nabla f(\gamma(t))\cdot\gamma'(t),\quad \forall t\in(-\epsilon,\epsilon).$$

Setting $t=0$, we find that $\nabla f(a)\cdot\gamma'(0)=0$. Repeating for all possible curves γ, we find that $\nabla f(a)$ is orthogonal to every tangent vector to S at a. $\square$

Remark. *By applying the chain rule to f composed instead with the straight line $\alpha(t)=a+tu$, for some unit vector $u\in\mathbb{R}^n$, we find that*

$$D_uf(a)=(f\circ\alpha)'(0)=\nabla f(\alpha(0))\cdot\alpha'(0)=\nabla f(a)\cdot u.$$

So the Chain Rule also gives an alternative proof of Lemma 92.

Implicit Differentiation

Consider the circle $\{x^2+y^2=1\}$. How do we find $\frac{dy}{dx}$ at the point $(\frac{3}{5},\frac{-4}{5})$?

Locally near $(\frac{3}{5},\frac{-4}{5})$, we find that $y^2=1-x^2$ and $y<0$ implies $y=-\sqrt{1-x^2}$ explicitly. So y *is* a function of x near $(\frac{3}{5},\frac{-4}{5})$. We could then differentiation this formula to find what we wanted.

Alternatively, rather than finding an explicit representation of y as a function of x locally, we could utilise the Chain Rule. That is, for the equation $x^2+y^2=1$, consider x as a variable and y as a function of x. Then, differentiating with respect to x, we have

$$2x+2y\frac{\partial y}{\partial x}=0.$$

Therefore, at the point $(\frac{3}{5},\frac{-4}{5})$ we have

$$2(\frac{3}{5})+2(\frac{-4}{5})\frac{\partial y}{\partial x}=0,$$

which simplifies to $\frac{\partial y}{\partial x}|_{(\frac{3}{5},\frac{-4}{5})}=\frac{3}{4}$.

Remark. *Unlike at the point* $(\frac{3}{5}, \frac{-4}{5})$*, near the point* $(-1, 0)$*,* y *is not a function of* x *locally. In fact, when we put* $(-1, 0)$ *into our implicit formula for* $\frac{\partial y}{\partial x}$ *where we assumed* y *was locally a function of* x*, we end up with the nonsense* $-2 + 0 = 0$*.*

Instead, x *is locally a function of* y *at the point* $(-1, 0)$*, and so we have the implicit equation* $2x\frac{\partial x}{\partial y} + 2y = 0$*, which results in* $\frac{\partial x}{\partial y}|_{(-1,0)} = 0$*.*

Example 107. Let $S = \{x^3 + z^2 + ye^{xz} + z\cos y = 0\}$. Given that z can be expressed locally about $(0, 0, 0)$ as a function of x and y, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$.

Differentiating our implicit formula for z as a function of x and y, we find

$$3x^2 + 2z\frac{\partial z}{\partial x} + ye^{xz}(z + x\frac{\partial z}{\partial x}) + \frac{\partial z}{\partial x}\cos y = 0$$
$$2z\frac{\partial z}{\partial y} + e^{xz} + ye^{xz}(x\frac{\partial z}{\partial y}) + \frac{\partial z}{\partial y}\cos y - z\sin y = 0$$

Setting $x = y = z = 0$, we find $\frac{\partial z}{\partial x}|_{(0,0,0)} = 0$, and $\frac{\partial z}{\partial y}|_{(0,0,0)} = -1$.

Week 5

5.1 Extremal Points

Definition 5.24. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in A$.*

(i) f is said to have a global maximum at a if

$$f(a) \geq f(x), \quad \forall x \in A.$$

(ii) f is said to have a local maximum at a if

$$f(a) \geq f(x), \quad \text{for any } x \in A \text{ near } a.$$

More precisely, $\exists \delta > 0$ such that $f(a) \geq f(x)$ for any $x \in A \cap B_\delta(a)$.

We make the exact same definitions for minimum values also.

Example 108. *Image of function from lectures to be inserted here*

This function has a

- Global maximum at 4,
- Global minimum at 1,
- Local maximum at 2 and 4,
- Local minimum at 1, 3 and 5.

The following examples show that global extrema need not exist.

Example 109. $f(x) = e^x$ for $x \in \mathbb{R}$. Since $\lim_{x\to\infty} f(x) = \infty$, there is no global maximum. Also, as $\lim_{x\to-\infty} f(x) = 0$ and $f(x) > 0$ for any $x \in \mathbb{R}$, then there is also no global minimum.

The domain $\mathbb{R}$ is not bounded.

Example 110. $f(x) = x$ for $x \in (-1, 1]$. There is a global maximum of 1, but f has no global minimum.

The domain $(-1, 1]$ is not closed.

Example 111. $f : [-1, 1] \to \mathbb{R}$, defined piecewise by

$$f(x) = \begin{cases} 1 - x & : x \in (0, 1] \\ 0 & : x = 0 \\ -1 - x & : x \in [-1, 0) \end{cases}.$$

f gets arbitrarily close to ± 1 but never reaches them, and so f has neither a global maximum or a global minimum.

The function f is not continuous.

The following theorem provides a sufficient condition, but not a necessary condition, for the existence of global extrema.

Theorem 112 (Extreme Value Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ be closed and bounded and $f : A \to \mathbb{R}$ be continuous. Then f has a global maximum and minimum.*

Remark. *Closed and bounded subsets $A \subseteq \mathbb{R}^n$ are also known as **compact** subsets.*
Although there is a more general definition of compact subsets in any topological space, in Euclidean space being compact is equivalent to being closed and bounded. For the interested reader, look up the Heine-Borel theorem.

Although we may use the Extreme Value Theorem (EVT) to verify the existence of global extrema, the question now turns to how one may locate them.

Example 113. *Image of function from lectures to be inserted here*

Firstly, since $A = [0, 4]$ is compact and f is continuous, by the EVT we know that f has a global maxima and minima.
Recall from single variable calculus, extrema can only occur at points x where

(i) $f'(x) = 0$ (i.e at $x = 1, 2$)

(ii) $f'(x)$ DNE (i.e at $x = 3$)

(iii) $x \in \partial A$ (i.e at $x = 0, 4$)

Points in cases (i) and (ii) are known as critical points and in case (iii) as boundary points.
Comparing the value of f at these five points, we find that f has a global max at 4 and a global min at 3.

We generalise the definition of a critical point to multi-variable functions.

Definition 5.25. $f : A \to \mathbb{R}$, $a \in \mathrm{Int}(A)$. *Then a is called a critical point of f if either:*

- $\nabla f(a)$ *DNE;*
- $\nabla f(a) = 0$.

Theorem 114 (First derivative test)**.** *If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ attains a local extremum at $a \in \operatorname{Int}(A)$, then a is a critical point of f.*

Proof. Since $a \in \operatorname{Int}(A)$, we can find $\delta > 0$ such that $a + tv \in A$ for any $|t| < \delta$ and any unit vector $v \in \mathbb{R}^n$.

Without loss of generality, we may assume that $\nabla f(a)$ exists. Then, for any $i \in \{1, \ldots, n\}$ consider the function

$$g_i(t) = f(a + te_i), \quad t \in (-\delta, \delta).$$

By the definition of g_i, we find that g_i has a local extremum at 0 and $g_i'(0) = \frac{\partial f}{\partial x_i}(a)$ exists. Therefore, by the first derivative test for single varaible functions

$$\nabla f(a) = \left(g_1'(0), \ldots, g_n'(0)\right) = (0, \ldots, 0) \in \mathbb{R}^n. \quad \square$$

Finding Extrema

We employ the following strategy for find extremal points:

1. Find critical points of our function in the interior $\operatorname{Int}(A)$.
2. Find the max/min of our function on the boundary ∂A.
3. Compare these values.

Example 115. $f(x, y) = x^2 + 2y^2 - x + 3$, defined for $(x, y) \in A = \overline{B_1(0)}$.

Firstly, we note that A is compact and since f is a polynomial it is continuous, so by the EVT f has global extrema in A.

1. $\nabla f(x, y) = (2x - 1, 4y)$ exists everywhere in $\operatorname{Int}(A) = B_1(0)$. Also, $\nabla f(x, y) = 0$ iff $(x, y) = (\frac{1}{2}, 0) \in B_1(0)$, with $f(\frac{1}{2}, 0) = \frac{11}{4}$.
2. We may parameterise ∂A by $\theta \mapsto (\cos\theta, \sin\theta)$, for $\theta \in [0, 2\pi]$. Plugging this into our function f we find that

$$\begin{aligned} f(\cos\theta, \sin\theta) &= \cos^2\theta + 2\sin^2\theta - \cos\theta + 3 \\ &= \cos^2\theta + 2(1 - \cos^2\theta) - \cos\theta + 3 \\ &= 5 - \cos\theta - \cos^2\theta \\ &= \frac{21}{4} - (\cos\theta + \frac{1}{2})^2. \end{aligned}$$

   From this, we can easily read off that the
   - maximum value on ∂A is $\frac{21}{4}$ when $\cos\theta = -\frac{1}{2}$, i.e at the points $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.
   - minimum value of ∂A is $\frac{21}{4} - \frac{3}{2}^2 = 3$ when $\cos\theta = 1$, i.e at the point $(1, 0)$.
3. Comparing our critial points and our boundary points, we find that

- $f(\frac{1}{2}, 0) = \frac{11}{4}$ is the global min,
- $f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{21}{4}$ is the global max.

Example 116. $f(x, y) = \sqrt{x^2 + y^4} - y$, defined for $(x, y) \in A = \{-1 \leq x, y \leq 1\}$.

A compact and f continuous implies that f has global extrema by the EVT.

1. Consider f restricted to $\mathrm{Int}(A) = \{-1 < x, y < 1\}$. At the origin we find that $\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{|h|-0}{h}$ DNE. Away from the origin f is differentiable with
$$\nabla f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^4}}, \frac{2y^3}{\sqrt{x^2 + y^4}} - 1\right),$$
and so $\nabla f = 0$ iff $(x, y) = (0, \frac{1}{2})$. Thus, f has two critical points $(0, 0)$ and $(0, \frac{1}{2})$ in $\mathrm{Int}(A)$, and $f(0, 0) = 0$, $f(0, \frac{1}{2}) = -\frac{1}{4}$.
2. We split ∂A into three components:
   - If $y = 1$, then $f(x, 1) = \sqrt{x^2 + 1} - 1$, which implies $f(x, 1) \in [0, \sqrt{2} - 1]$.
   - If $y = -1$, then $f(x, -1) = \sqrt{x^2 + 1} + 1$, which implies $f(x, -1) \in [2, \sqrt{2} + 1]$.
   - If $x = \pm 1$, then $f(\pm 1, y) = \sqrt{1 + y^4} - y$, which implies $f(\pm 1, y) \in (0, \sqrt{2} + 1]$.

   Therefore, on ∂A, the minimum value is 0 at $(0, 1)$, and the maximum value is $\sqrt{2} + 1$ at $(\pm 1, -1)$.
3. $f(0, \frac{1}{2}) = -\frac{1}{4}$ is a global minimum, and $f(\pm 1, -1) = 1 + \sqrt{2}$ is a global maximum.

Unbounded Regions

In practice, we may wish to find extremal points for functions defined on unbounded regions. Although we can't directly apply the EVT, in many cases we may still use it on a suitably chosen compact subset our unbounded region.

Example 117. $f(x, y) = x^2 + y^2 - 4x + 6y + 7$ for all $(x, y) \in \mathbb{R}^2$. Firstly, we observe that $\lim_{(x,y)\to\infty} f(x, y) = \infty$, and so f has no global maximum on $\mathbb{R}^2$. As we shall see, f will have a global minimum.

The gradient $\nabla f(x, y) = (2x - 4, 2y + 6)$ exists on all of $\mathbb{R}^2$, with $\nabla f = 0$ iff $(x, y) = (2, -3)$. That is, there is only one critical point in $\mathbb{R}^2$, with $f(2, -3) = -6$.

We want to chose a $A \subseteq \mathbb{R}^2$ compact such that $(2, -3) \in \mathrm{Int}(A)$, but also so that outside of $\mathrm{Int}(A)$, f is larger than -6.

Note that

$$\begin{aligned} f(x,y) &= x^2 + y^2 - 4x - 6y + 7 \\ &\geq x^2 + y^2 - 4(\sqrt{x^2+y^2}) - 6(\sqrt{x^2+y^2}) + 7 \\ &\geq (\sqrt{x^2+y^2})((\sqrt{x^2+y^2} - 10) + 7, \end{aligned}$$

and so if $\sqrt{x^2+y^2} \geq 10$, $f(x,y) \geq 7$. Therefore, we chose $A = \overline{B_{10}(0)}$.

By the EVT, f restricted to A has a global minimum in A. Since there is only one critical point $(2,-3)$ in $\text{Int}(A)$, and $f \geq 7 > -6$ on ∂A, it follows that f has a minimum at $(2,-3)$ in A. Finally, since $f(x,y) \geq 7 > f(2,-3)$ for any $(x,y) \notin A$, f does indeed have a global minimum at $(2,-3)$.

Example 118. Suppose you are tasked with constructing a box (without a lid) that has a volume of $16m^3$. The material used to construct the base costs $\$2/m^2$, and the material for the sides costs $\$0.5/m^2$.

Q: What is the cheapest you can build such a box?

Suppose $x, y > 0$ denote the length (in m) of the sides of the base. It follows that the height of the box must then by $\frac{16}{xy}$. With these dimensions, the cost of the box is given by the function $C : \Omega = \{x, y > 0\} \to \mathbb{R}$,

$$\begin{aligned} C(x,y) &= 2xy + \frac{1}{2}(2\frac{16}{xy}y + 2\frac{16}{xy}x) \\ &= 2xy + \frac{16}{x} + \frac{16}{y}. \end{aligned}$$

We have formulated our problem as finding a global minimum of the function C on the unbounded domain Ω.

1. $\nabla C(x,y) = (2y - \frac{16}{x^2}, 2x - \frac{16}{y^2})$ exists everywhere in Ω, and $\nabla C = 0$ iff $(x,y) = (2,2)$. That is, C has one critical point $(2,2) \in \Omega$, with $C(2,2) = 24$.

2. We now find a compact region $A \subseteq \Omega$ such that $(2,2) \in \text{Int}(A)$ and $C > 24$ outside of $\text{Int}(A)$. We choose $A = [\frac{1}{10}, 1000] \times [\frac{1}{10}, 1000]$.

   If $x \leq \frac{1}{10}$ or $y \leq \frac{1}{10}$, then $C(x,y) > \frac{16}{x} + \frac{16}{y} \geq 160$. Alternatively, if either $x \geq \frac{1}{10}$ and $y \geq 1000$, or $x \geq 1000$ and $y \geq \frac{1}{10}$, then $xy \geq 100$ and $C(x,y) > 2xy \geq 200$.

3. Since A is compact and C is continuous, by the EVT, C has a global minimum in A. Since $(2,2) \in \text{Int}(A)$, and $C > 24 = C(2,2)$ on ∂A, it follows that C has a minimum at $(2,2)$ in A. Finally, since $C > 24$ at every point in $\Omega \setminus A$, C has a global minimum at $(2,2)$ in Ω, and the cheapest price to construct the box is $\$24$.

5.2 Taylor Series

Let $g : \mathbb{R} \to \mathbb{R}$ be a function of one-variable. The Taylor expansion of $g(t)$ at $t = 0$ up to order $k \in \mathbb{N}$ is

$$g(t) = g(0) + g'(0)t + \frac{1}{2!}g''(0)t^2 + \cdots + \frac{1}{k!}g^{(k)}(0)t^k + \varepsilon_k(t), \tag{5.18}$$

where ε_k is the remainder and satisfies $\lim_{t\to 0} \frac{\varepsilon_k(t)}{t^k} = 0$. We want a similar formula for a multi-variable function.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $x, a \in \mathbb{R}^n$. Restricting f to the line $L = \{a + t(x - a) : t \in \mathbb{R}\}$ we have the single variable function $g(t) = f(a + t(x - a))$. Applying (5.18) to this function g, we find that

$$f(x) = g(1) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \cdots + \frac{1}{k!}g^{(k)}(0) + \varepsilon_k(1).$$

We now find expressions for derivatives of g at $t = 0$ in terms of f:

- $g(0) = f(a)$,
- By the Chain Rule, $g'(t) = \nabla f(a + t(x - a)) \cdot (x - a)$, and so

$$g'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$

- Again by the Chain Rule,

$$g''(t) = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(a + t(x - a))\right)(x_i - a_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a+t(x-a))(x_j - a_j)(x_i - a_i),$$

  and so

$$g''(0) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)(x_j - a_j)(x_i - a_i).$$

- Repeating, we find that for any $k \in \mathbb{N}$,

$$g^{(k)}(0) = \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k}).$$

Definition 5.26. *Let $\Omega \subseteq \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$ be a C^k function, for some $k \in \mathbb{N}$. Then, for any $a \in \Omega$, define the k^{th}-order Taylor polynomial of f at a to be*

$$P_k(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \cdots + \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k}).$$

Theorem 119 (Taylor's Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$ be a C^k function, for some $k \in \mathbb{N}$. Then, for any $x, a \in \Omega$, we have*

$$f(x) = P_k(x) + \varepsilon_k(x),$$

with $\lim_{x\to a} \frac{\varepsilon_k(x)}{\|x-a\|^k} = 0$.

Remark.
- *$P_1(x)$ is the best affine approximation of f at a, with $\varepsilon_1(x)$ denoting the usual error function.*
- *The partial derivatives of P_k and f agree up to order k at a.*

Example 120. Suppose $f(x, y)$ is a C^2 function, then its 2^{nd}-order Taylor polynomial at (x_0, y_0) is given by

$$\begin{aligned} P_2(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &\quad + \frac{1}{2}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + \underbrace{2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)}_{\text{Clairaut's theorem}} + f_{yy}(x_0,y_0)(y-y_0)^2 \right). \end{aligned}$$

Example 121. $f(x, y) = e^x \cos y$. Then the partial derivatives of f are

$$f_x = e^x \cos y, \quad f_y = -e^x \sin y,$$

$$f_{xx} = e^x \cos y, \quad f_{xy} = f_{yx} = -e^x \sin y, \quad f_{yy} = -e^x \cos y.$$

At $(0, 0)$, we have

$$f = f_x = f_{xx} = 1, \quad f_y = f_{xy} = f_{yx} = 0, \quad f_{yy} = -1,$$

and therefore at $(0, 0)$, the 2^{nd}-order Taylor polynomial is

$$P_2(x,y) = 1 + x + \frac{x^2}{2} - \frac{y^2}{2}.$$

Taking more partial derivatives of f, we see that

$$f_{xxx} = e^x \cos y, \quad f_{xxy} = f_{xyx} = f_{yxx} = -e^x \sin y, \quad f_{xyy} = f_{yxy} = f_{yyx} = -e^x \cos y, \quad f_{yyy} = e^x \sin y,$$

and so at $(0, 0)$, we have

$$f_{xxx} = 1, \quad f_{xxy} = f_{xyx} = f_{yxx} = f_{yyy} = 0, \quad f_{xyy} = f_{yxy} = f_{yyx} = -1,$$

and therefore at $(0, 0)$, the 3^{rd}-order Taylor polynomial is

$$\begin{aligned} P_3(x,y) &= P_2(x,y) + \frac{g^{(3)}(0)}{6!} \\ &= 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2}. \end{aligned}$$

Hessian Matrix

Definition 5.27. *Let $\Omega \subseteq \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$ a C^2 function. The Hessian matrix of f at $a \in \Omega$ is*

$$Hf(a) := \begin{pmatrix} f_{x_1x_1}(a) & \cdots & f_{x_1x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(a) & \cdots & f_{x_nx_n}(a) \end{pmatrix}.$$

Remark. *By Clairaut's theorem, the Hessian matrix is a symmetric $n \times n$ matrix.*

Using $Hf(a)$, we can rewrite the 2^{nd}-order Taylor polynomial:

$$P_2(x) = \underbrace{f(a)}_{1\times 1} + \underbrace{\nabla f(a)}_{1\times n}\underbrace{(x-a)}_{n\times 1} + \frac{1}{2}\underbrace{(x-a)^T}_{1\times n}\underbrace{Hf(a)}_{n\times n}\underbrace{(x-a)}_{n\times 1}.$$

Example 122. Considering the same function as before $f(x, y) = e^x \cos y$, we had

$$f(0,0) = 1, \quad \nabla f(0,0) = (1,0), \quad Hf(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$P_2(x,y) = 1 + (1,0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Example 123. Find $P_2(x, y)$ at the point $(1, 0)$ for the function $g(x, y) = \frac{\log x}{1-y}$.

$$\begin{aligned} g(1,0) &= 0 \\ \nabla g &= \left(\frac{1}{x(1-y)}, \frac{\log x}{(1-y)^2}\right), \quad \nabla g(1,0) = (1,0) \\ Hg &= \begin{pmatrix} \frac{-1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{2\log x}{(1-y)^3} \end{pmatrix}, \quad Hg(1,0) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} P_2(x,y) &= g(1,0) + \nabla g(1,0) \begin{pmatrix} x-1 \\ y \end{pmatrix} + \frac{1}{2}(x-1,y)Hg(1,0)\begin{pmatrix} x-1 \\ y \end{pmatrix} \\ &= 0 + \nabla(1,0)\begin{pmatrix} x-1 \\ y \end{pmatrix} + \frac{1}{2}(x-1,y)\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y \end{pmatrix} \\ &= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y. \end{aligned}$$

5.3 Second Derivative Test

Suppose $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, and $a \in \Omega$ is a critical point of f. Then $\nabla f(a) = 0$. For x near a, we have that $f(x) \approx P_2(x)$. That is,

$$\begin{aligned} f(x) - f(a) &\approx P_2(x) - f(a) \\ &= \underbrace{\nabla f(a)}_{=0}(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a) \\ &= \frac{1}{2}(x-a)^T Hf(a)(x-a). \end{aligned}$$

Therefore, the Hessian matrix can potentially determine whether a is a local extremum.

For $n = 1$, $\frac{1}{2}(x-a)^T Hf(a)(x-a) = \frac{1}{2}f''(a)(x-a)^2$, and we see that

$$\begin{cases} f''(a) > 0 & \implies \text{local min at } a, \\ f''(a) < 0 & \implies \text{local max at } a. \end{cases}$$

For $n = 2$, lettting $a = (x_0, y_0)$, we have

$$\frac{1}{2}(x-a)^T Hf(a)(x-a) = \frac{1}{2}(x-x_0, y-y_0)\begin{pmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}.$$

To understand the nature of such critical points, we study quadratic forms (in this case of two variables)

$$q(x,y) = (x,y)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = Ax^2 + 2Bxy + Cy^2.$$

Q: Does $q(x,y)$ have a sign?

Example 124. If $q(x,y) = 2xy$, we could rewrite this as

$$q(x,y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2.$$

What is the sign of $q(x,y)$ away from the origin $(0,0)$?

- Along $x + y = 0$, $q(x,-x) = -2x^2 < 0$.
- Along $x - y = 0$, $q(x,x) = 2x^2 > 0$.

Therefore, q is indefinite. Clearly, the origin is a critical point of $q(x,y)$, but neither a local minimum or a local maximum. We call such a point a saddle point.

Example 125. If $q(x,y) = 17x^2 - 12xy + 8y^2$, we could again complete the square

$$q(x,y) = 17(x - \frac{6}{17}y)^2 + \frac{100}{17}y^2. \tag{5.19}$$

In this form, it is then clear that $q(x,y) > 0 = q(0,0)$ for any $(x,y) \neq (0,0)$, and the critical point at the origin is a local (and global) minimum of q.

Remark. *An expression such as (5.19) is known as a diagonalisation of q, and is not unique. For example, we could also write our quadratic form q as*

$$q(x,y) = 5\left(\frac{x+2y}{\sqrt{5}}\right)^2 + 20\left(\frac{2x-y}{\sqrt{5}}\right)^2.$$

Different diagonalisations are related to one another by means of an orthogonal change of coordinates.

Example 126. Consider the quadratic form $q(x,y,z) = xy + yz + xz$ of three variables. By completing the square twice, we rewrite q in the form

$$\begin{aligned} q(x,y,z) &= xy + yz + xz \\ &= \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y) \\ &= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2. \end{aligned}$$

- Along the line $x - y = z = 0$, $q(x,x,0) = x^2 > 0$.
- On the plane $x + y + 2z = 0$, $q(x,y,-\frac{1}{2}(x+y)) = -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2 < 0$.

Therefore, the critical point $(0,0,0)$ is a saddle point.

Definition 5.28. *Let A be an $n \times n$ symmetric matrix. Then A is said to be*

1. *positive definite if $x^T Ax > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$;*
2. *negative definite if $x^T Ax < 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$;*
3. *indefinite if $x^T Ax > 0$ and $y^T Ay < 0$, for some $x, y \in \mathbb{R}^n$.*

Remark. *Not every symmetric $n \times n$ matrix fits into one of these three cases.*

Example 127. The matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ is positive definite, $B = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$ is negative definite, $C = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$ is indefinite, and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is neither positive definite, negative definite, or indefinite.

The following theorem, known as the Second Derivative Test, determine the nature of a critical point via the definiteness of the Hessian matrix at that point.

Theorem 128 (Second Derivative Test)**.** *If $\Omega \subseteq \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}$ is a C^2 function, and $a \in \Omega$ is a critical point of f. Then $Hf(a)$ is*

(i) positive definite $\implies$ a is a local minimum;

(ii) negative definite $\implies$ a is a local maximum;

(iii) indefinite $\implies$ a is a saddle point.

Proof. Without loss of generality, we may assume that $a = 0$. By Taylor's theorem, since $\nabla f(0) = 0$, we have that

$$f(x) - f(0) = \frac{1}{2}x^T Hf(0)x + \varepsilon(x), \quad \forall x \in \Omega,$$

with $\lim_{x\to 0} \frac{\varepsilon(x)}{\|x\|^2} = 0$. Next, consider the map $F : \partial B_1(0) \subseteq \mathbb{R}^n \to \mathbb{R}$, defined by

$$F(x) := \frac{1}{2}x^T Hf(0)x, \quad \forall \text{ unit vectors } x \in \partial B_1(0).$$

If $Hf(0)$ is positive definitie, then $F > 0$. Moreover, since $\partial B_1(0)$ is compact and F is continuous (F is a polynomial of degree 2), by the EVT this map has a global minimum. In particular, there exists a unit vector $x_0 \in \mathbb{R}^n$ such that

$$F(x) \geq F(x_0) > 0, \quad \forall x \in \partial B_1(0).$$

Setting $\epsilon_0 := F(x_0) > 0$, we find that for any non-zero vector $x \in \mathbb{R}^n$

$$\frac{1}{2}x^T Hf(0)x = \frac{1}{2}\left(\frac{x}{\|x\|}\right)^T Hf(0)\left(\frac{x}{\|x\|}\right)\|x\|^2 \geq \epsilon_0\|x\|^2.$$

Plugging this choice of ϵ_0 into the definition of $\lim_{x\to 0} \frac{\varepsilon(x)}{\|x\|^2} = 0$, we find $\delta > 0$ such that, if $x \in \Omega$ and $0 < \|x\| < \delta$, then $\frac{|\varepsilon(x)|}{\|x\|^2} < \epsilon_0$. In particular, at any point $x \in \Omega$ with $0 < \|x\| < \delta$,

$$\begin{aligned}
f(x) - f(0) &= \frac{1}{2}x^T Hf(0)x + \varepsilon(x) \\
&\geq \left|\frac{1}{2}x^T Hf(0)x\right| - |\varepsilon(x)| \\
&\geq \epsilon_0\|x\|^2 - |\varepsilon(x)| \\
&> \epsilon_0\|x\|^2 - \epsilon_0\|x\|^2 = 0,
\end{aligned}$$

and hence 0 is a local minimum of f. The same argument applies if $Hf(0)$ is negative definite. Finally, in the case that $Hf(0)$ is indefinite, we find unit vectors $x, y \in \mathbb{R}^n$ such that

$$\alpha := \frac{1}{2}x^T Hf(0)x > 0, \quad \beta := \frac{1}{2}y^T Hf(0)y < 0.$$

Consider the sequence of vectors $x_j := \frac{x}{j} \in \mathbb{R}^n$. Note that $x_j \to 0$ as $j \to \infty$. Moreover,

$$\begin{aligned}
\lim_{j\to\infty}\left[j^2(f(x_j) - f(0))\right] &= \lim_{j\to\infty} j^2\left[\frac{1}{2}x_j^T Hf(0)x_j + \varepsilon(x_j)\right] \\
&= \frac{1}{2}x^T Hf(0)x + \lim_{j\to\infty}\frac{\varepsilon(x_j)}{\|x_j\|^2} \\
&= \alpha + 0 = \alpha > 0.
\end{aligned}$$

Therefore, for j sufficiently large, $f(x_j) > f(0)$. Repeating the argument for the sequence $y_j = \frac{y}{j}$, we find that for j sufficiently large, $f(y_j) < f(0)$. Thus we conclude that 0 is a saddle point. □

In practise, how do we determine if the Hessian matrix is definite?

Lemma 129. *Let* $M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ *be a* 2×2 *matrix. Then*

- *M is positive definite* $\iff$ $A > 0$*, and* $\det(M) > 0$*;*
- *M is negative definite* $\iff$ $A < 0$*, and* $\det(M) > 0$*;*
- *M is indefinite* $\iff$ $\det(M) < 0$.

Proof. Let $q(x, y) = Ax^2 + 2Bxy + Cy^2$.

(i) We begin with the case that $A \neq 0$:

$$Aq(x, y) = A^2x^2 + 2ABxy + ACy^2 = (Ax + By)^2 + \underbrace{(AC - B^2)}_{\det M} y^2.$$

From this we conclude that $Aq > 0$ if $\det(M) > 0$, and Aq changes sign if $\det(M) < 0$. Dividing by A gives us that

- $q > 0 \iff A > 0, \det(M) > 0$;
- $q < 0 \iff A < 0, \det(M) > 0$;
- q changes sign $\iff \det(M) < 0$.

(ii) The case $A = 0$:

$$q(x, y) = y(2Bx + Cy),$$

is clearly neither positive definite nor negative definite. q is indefinite $\iff B \neq 0 \iff \det(M) = -B^2 < 0$. □

We use this Lemma to update the Second Derivative Test in the case $n = 2$.

Theorem 130. *If* $\Omega \subseteq \mathbb{R}^2$ *is open,* $f : \Omega \to \mathbb{R}$ *is a* C^2 *function, and* $a \in \Omega$ *is a critical point of* f*. Then*

(i) $f_{xx}f_{yy} - f_{xy}^2 > 0, f_{xx} > 0$ *at* $a \implies$ *a is a local minimum;*

(ii) $f_{xx}f_{yy} - f_{xy}^2 > 0, f_{xx} < 0$ *at* $a \implies$ *a is a local maximum;*

(iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ *at* $a \implies$ *a is a saddle point;*

(iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ *at* $a \implies$ *inconclusive.*

Remark. *Case (iv) could be a local max/min or a saddle point (see Example 133).*

Example 131. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$. This is a smooth function with

$$\nabla f(x, y) = (6x - 10y + 2, -10x + 6y + 2),$$

and so the only critical point of f is at $(x, y) = (\frac{1}{2}, \frac{1}{2})$. The Hessian matrix of f is

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6 & -10 \\ -10 & 6 \end{pmatrix}.$$

Note that (at the critical point)

$$f_{xx} f_{yy} - f_{xy}^2 = 6^2 - 10^2 = -64 < 0,$$

and by the Second Derivative Test, $(\frac{1}{2}, \frac{1}{2})$ is a saddle point.

Example 132. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 3x - x^3 - 3xy^2$. This is a smooth function with

$$\nabla f(x, y) = (3 - 3x^2 - 3y^2, -6xy),$$

and so the critical points of f are when

$$xy = 0, x^2 + y^2 = 1 \iff (x, y) = (0, \pm 1), (\pm 1, 0).$$

The Hessian matrix of f is

$$Hf(x, y) = \begin{pmatrix} -6x & -6y \\ -6y & -6x \end{pmatrix}.$$

Therefore, at each critical point we deduce that:

- $Hf(1, 0) = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$, with $\det(Hf(1, 0)) = 36 > 0$ and $f_{xx} = -6 < 0$. So $(1, 0)$ is a local maximum.
- $Hf(-1, 0) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, with $\det(Hf(-1, 0)) = 36 > 0$ and $f_{xx} = 6 > 0$. So $(-1, 0)$ is a local minimum.
- $Hf(0, 1) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$, with $\det(Hf(0, 1)) = -36 < 0$. So $(0, 1)$ is a saddle point.
- $Hf(0, -1) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$, with $\det(Hf(0, -1)) = -36 < 0$. So $(0, -1)$ is a saddle point.

Example 133. Consider the three functions

$$f(x, y) = x^2 + y^4, \quad g(x, y) = x^2 - y^4, \quad h(x, y) = -x^2 - y^4.$$

Since

$$\nabla f(x, y) = (2x, 4y^3), \quad \nabla g(x, y) = (2x, -4y^3), \quad \nabla h(x, y) = (-2x, -4y^3),$$

all three functions have a single critical point at $(0, 0)$. Since

$$Hf(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix}, \quad Hg(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & -12y^2 \end{pmatrix}, \quad Hh(x, y) = \begin{pmatrix} -2 & 0 \\ 0 & -12y^2 \end{pmatrix},$$

and

$$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad Hg(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad Hh(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

all three Hessian matrices have zero determinant at the origin, and the Second Derivative Test is inconclusive. However at $(0,0)$, we see that f has a local minimum, g has a saddle point, and h has a local maximum.

Let us return to the general case $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, $a \in \Omega$ and a is a critical point of f. Then

$$Hf(a) = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}.$$

Since f is C^2, $Hf(a)$ is symmetric, and thus by a theorem from Linear Algebra, there exists an orthogonal matrix P such that

$$P^T Hf(a) P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

is a diagonal matrix, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are the eigenvalues of $Hf(a)$.

(Recall, a matrix P is called orthogonal if $P^T P = I_n$.)

Note that, the diagonal matrix $P^T Hf(a) P$ is

- positive definite $\iff$ $\lambda_i > 0$ for any $i \in \{1, \dots, n\}$;
- negative definite $\iff$ $\lambda_i < 0$ for any $i \in \{1, \dots, n\}$;
- indefinite $\iff$ $\lambda_i < 0 < \lambda_j$ for some $i, j \in \{1, \dots, n\}$.

Since P is invertible ($P^{-1} = P^T$), we see that the previous criteria also apply to the matrix $Hf(a)$.

Alternatively, we can also check the definiteness of the Hessian matrix by looking at submatrices. That is, let H_k denote the $k \times k$ submatrix

$$H_k = \begin{pmatrix} f_{11} & \cdots & f_{1k} \\ \vdots & \ddots & \vdots \\ f_{k1} & \cdots & f_{kk} \end{pmatrix}, \qquad \text{for } k \in \{1, \dots, n\}.$$

Then, $Hf(a)$ is

- positive definite $\iff$ $\det(H_k) > 0$, for $k \in \{1, \dots, n\}$;
- negative definite $\iff$ $\operatorname{sign}(\det(H_k)) = (-1)^k$, for $k \in \{1, \dots, n\}$.

For $n = 2$,

$$\det(H_1) = f_{xx}, \quad \det(H_2) = f_{xx} f_{yy} - f_{xy}^2,$$

and we recover the $n = 2$ version of the Second Derivative Test stated earlier.

Week 6

6.1 Lagrange Multipliers

To motivate the next section, we consider the following question:

what if we want extrema subject to constraints?

Theorem 134. *Let $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^1 functions, $c \in \mathbb{R}$, and*

$$S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}.$$

If a is a local extremum of f on S, and $\nabla g(a) \neq 0$, then $\nabla f(a) = \lambda \nabla g(a)$ for some $\lambda \in \mathbb{R}$.

Remark. *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ denote the function $F(x, \lambda) = f(x) - \lambda(g(x) - c)$. Then*

$$\nabla F(x, \lambda) = (\nabla f(x) - \lambda \nabla g(x), c - g(x)).$$

Therefore, critical points of f under the constraint $g \equiv c$ are equivalent to unconstrained critical points of F.

Example 135. Suppose we want to find the point on the parabola $4y = x^2$ closest to the point $(1, 2)$.

We may rephrase this by introducing the squared distance function $f(x, y) = (x-1)^2 + (y-2)^2$, and the constraining function $g(x, y) = x^2 - 4y$. Then our question becomes find the minimum of f on the level set $g = 0$.

Since f, g are polynomials on $\mathbb{R}^2$ they are certainly C^1 on $\mathbb{R}^2$ with

$$\nabla f = (2(x-1), 2(y-2)), \quad \nabla g = (2x, -4) \neq 0.$$

Thus, if (x, y) is a local extremum of f on $g = 0$, then

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) & : \text{ for some } \lambda \in \mathbb{R} \\ g(x, y) = 0, \end{cases}$$

which is equivalent to the system of equations

$$2(x-1) = 2\lambda x, \quad 2(y-2) = -4\lambda, \quad x^2 = 4y.$$

Solving this system, we find a unique solution $(x, y) = (2, 1)$.

Geometrically, f must have a minimum on $g = 0$. By our reasoning, $(2, 1)$ is the only possible candidate, and thus f has a minimum at $(2, 1)$ on the level set $g \equiv 0$. Since $f(2, 1) = 2$, in the original language of the question, the point $(2, 1)$ is the point lying on the parabola $4y = x^2$ lying closest to $(1, 2)$, with a distance of $\sqrt{2}$ between the points.

Idea of Proof. Suppose f has an extremum on S at a. Let $\gamma : (\epsilon, \epsilon) \to S$ be a curve on S such that $\gamma(0) = a$. Since $f \circ \gamma$ has an extremum at 0,

$$0 = (f \circ \gamma)'(0) = \nabla f(a) \cdot \gamma'(0),$$

and so $\nabla f(a) \perp S$ at a. Since $\nabla g(a)$ is also normal to S at a, we find that $\nabla f(a)$ and $\nabla g(a)$ are parallel. $\square$

Example 136. Find the point on the parabola $x^2 = 4y$ closest to the point $(2, 5)$. As before, we set up the problem in the following way:

Minimise $f(x, y) = (x - 2)^2 + (y - 5)^2$ with the constraint $g(x, y) = x^2 - 4y = 0$.

As in the previous example, f, g are C^1 functions with

$$\nabla f = (2(x - 2), 2(y - 5)), \quad \nabla g = (2x, -4) \neq 0.$$

Thus, if (x, y) is a local extremum of f on $g = 0$, then

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) & : \text{ for some } \lambda \in \mathbb{R} \\ g(x, y) = 0, \end{cases}$$

which is equivalent to the system of equations

$$2(x - 2) = 2\lambda x, \quad 2(y - 5) = -4\lambda, \quad x^2 = 4y.$$

Solving this system, we now find two solutions $(x, y) = (-2, 1), (4, 4)$.

Again, geometrically, f must have a minimum on $g = 0$. Since $f(-2, 1) = 32$ and $f(4, 4) = 5$, f has a minimum at $(4, 4)$ on the level set $g \equiv 0$, and the point $(4, 4)$ is the point lying on the parabola $4y = x^2$ lying closest to $(2, 5)$.

Example 137. Maximise the function $f(x, y) = xy^2$ on the ellipse $g(x, y) = x^2 + 4y^2 = 4$.

Since f is continuous and the level set $\{g = 4\}$ is compact, by the EVT f has a global maximum on the ellipse. Since f, g are C^1 functions with

$$\nabla f = (y^2, 2xy), \quad \nabla g = (2x, 8y),$$

and $\nabla g \neq 0$ on the ellipse, we may use the theory of Lagrange Multipliers to conclude that at such a maximum

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) & : \text{ for some } \lambda \in \mathbb{R} \\ g(x, y) = 4. \end{cases}$$

This is equivalent to the system of equations

$$y^2 = 2\lambda x, \quad 2xy = 8\lambda y, \quad x^2 + 4y^2 = 4.$$

If $y = 0$, then $x = \pm 2$. Otherwise $y \neq 0$, and from our equations we deduce that

$$\frac{2xy}{y} = \frac{8\lambda y}{2\lambda x} \implies x^2 = 2y^2,$$

and hence $(x, y) = (\pm\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}})$. Comparing our function at these six points, we find that

$$f(\pm 2, 0) = 0, \quad \underbrace{f\left(\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{4}{3\sqrt{3}}}_{\text{Maximum}}, \quad f\left(-\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{4}{3\sqrt{3}}.$$

For problems in finding unconstrained extremal points for functions $f : A \to \mathbb{R}$, the theory of Lagrange Multipliers can be utilised to determine the extremal points of f on the boundary ∂A.

Example 138. In Example 115, we found that $f(x, y) = x^2 + 2y^2 - x + 3$ has the single critical point $(\frac{1}{2}, 0) \in \operatorname{Int}(A) = B_1(0)$, with $f(\frac{1}{2}, 0) = \frac{11}{4}$. Instead of parameterising the boundary ∂A, we instead define $g(x, y) = x^2 + y^2$ and note that $\partial A = \{g = 1\}$. Since $\nabla g = (2x, 2y) \neq 0$ on ∂A, we may apply the theory of Lagrange Multipliers to deduce that at an extrema of f restricted to ∂A, we must satisfy the system of equations

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) & : \text{ for some } \lambda \in \mathbb{R} \\ g(x, y) = 1, \end{cases}$$

which is equivalent to

$$2x - 1 = 2\lambda x, \quad 4y = 2\lambda y, \quad x^2 + y^2 = 1.$$

From the second equation, we deduce that either $y = 0$, in which case $x = \pm 1$, or $\lambda = 2$, in which case $(x, y) = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. Comparing f at these points

$$\underbrace{f(\frac{1}{2}, 0) = \frac{11}{4}}_{\text{Minimum}}, \quad f(1, 0) = 3, \quad f(-1, 0) = 5, \quad , \underbrace{f\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{21}{4}}_{\text{Maximum}}.$$

The following theorem generalises to the case of multiple constraints.

Theorem 139. *Let $f, g_1, \dots, g_k : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^1 functions, $c \in \mathbb{R}^k$, and*

$$S = \{x \in \Omega : g_i(x) = c_i, \ i \in \{1, \dots, k\}\}.$$

If a is a local extremum of f on S, and $\nabla g_1(a), \dots, \nabla g_k(a)$ are linearly independent, then

$$\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a), \quad \text{for some } \lambda \in \mathbb{R}^k.$$

Idea of proof. Just like in the case of a single constraint, $\nabla f(a) \perp S$ at a. That is, $\nabla f(a)$ is perpendicular to the $(n - k)$-dimensional tangent plane to S at a. Therefore, $\nabla f(a)$ must belong to the span of the normal vectors $\nabla g_1(a), \dots, \nabla g_k(a)$. □

Example 140. Maximise the function $f(x,y,z) = x^2 + 2y^2 - z^2$ on the line L given by the two equations $2x - y = 0$, $y + z = 0$, assuming that we know f does admit a maximum on L.

Defining our constraints $g_1(x,y,z) = 2x - y$, $g_2(x,y,z) = y + z$, we find that

$$\nabla f = (2x, 2, -2z), \quad \nabla g_1 = (2, -1, 0), \quad \nabla g_2 = (0, 1, 1),$$

with $\nabla g_1(x,y,z), \nabla g_2(x,y,z)$ linearly independent at every point $(x,y,z) \in \mathbb{R}^3$. Therefore, we may use the theory of Lagrange Multipliers to deduce that at the maximum of f on L, we satisfy the system of equations

$$\begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) & : \text{for some } \lambda_1, \lambda_2 \in \mathbb{R} \\ g_1(x,y,z) = 0, \\ g_2(x,y,z) = 0, \end{cases}$$

which is equivalent to the system of equations

$$2x = 2\lambda_1, \quad 2 = -\lambda_1 + \lambda_2, \quad -2z = \lambda_2, \quad 2x - y = 0, \quad y + z = 0.$$

This system has the unique solution $(x,y,z) = (\frac{2}{3}, \frac{4}{3}, -\frac{4}{3})$. Since a maximum exists, it must occur at this point.

Example 141. Find the distance between the curve $xy = 1$ and the line $x + 4y = \frac{15}{8}$ in $\mathbb{R}^2$.

Given a point (x,y) on the curve and a point (u,v) on the line, the distance squared between them is given by the function $f(x,y,u,v) = (x-u)^2 + (y-v)^2$. The point (x,y) lying on the curve can be expressed as the constraint $g_1(x,y,u,v) = xy = 1$, and similarly $g_2(x,y,u,v) = u + 4v + \frac{15}{8}$ is equivalent to (u,v) lying on the line. Therefore, we need to minimise f under the constraints $g_1 = 1$, $g_2 = \frac{15}{8}$. Since

$$\begin{aligned} \nabla f &= (2(x-u), 2(y-v), -2(x-u), -2(y-v)) \\ \nabla g_1 &= (y, x, 0, 0) \\ \nabla g_2 &= (0, 0, 1, 4), \end{aligned}$$

$\nabla g_1, \nabla g_2$ are linearly independent away from the origin $(0,0)$. But $xy = 1$, so these vectors are linearly independent when our constraints are satisfied, and hence by the theory of Lagrange Multipliers, we know that at a minimum, we solve the system of equations

$$\begin{cases} \nabla f(x,y,u,v) = \lambda_1 \nabla g_1(x,y,u,v) + \lambda_2 \nabla g_2(x,y,u,v) & : \text{for some } \lambda_1, \lambda_2 \in \mathbb{R} \\ g_1(x,y,u,v) = 1, \\ g_2(x,y,u,v) = \frac{15}{8}. \end{cases}$$

This system of equations has two solutions: $(2, \frac{1}{2}, \frac{15}{8}, 0)$, and $(-2, -\frac{1}{2}, -\frac{225}{136}, \frac{15}{17})$. Since

$$\begin{aligned} f(2, \frac{1}{2}, \frac{15}{8}, 0) &= \left(\frac{1}{8}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{17}{64}, \\ f(-2, -\frac{1}{2}, -\frac{225}{136}, \frac{15}{17}) &= \underbrace{\left(2 - \frac{225}{136}\right)^2}_{\geq 0} + \underbrace{\left(\frac{15}{17} + \frac{1}{2}\right)^2}_{\geq 1} \geq 1 > \frac{17}{64}, \end{aligned}$$

f attains its minimum at $(2, \frac{1}{2}, \frac{15}{8}, 0)$, and the distance between the curve and the line is

$$\sqrt{f(2, \frac{1}{2}, \frac{15}{8}, 0)} = \frac{\sqrt{17}}{8}.$$

6.2 Implicit Function Theorem

We now move away from finding extremal points, and instead ask the following:

when can we 'solve' a constraint?

For example, if we have an implicit relation between two variables $g(x, y) = c$, when can we find, *locally*, an explicit formula $y = f(x)$ such that $g(x, f(x)) = c$?

Example 142. Consider the level set $g(x, y) = x^2 - y^2 = 0$. Near the point:

- $(1, 1)$, we find $y = x$;
- $(-1, 1)$, we find $y = -x$;
- $(0, 0)$, y is not uniquely determined by x.

Example 143. Consider the constraint $x^2 + y^2 + z^2 = 2$ in $\mathbb{R}^3$, and consider the point $(x, y, z) = (0, 1, 1)$ satisfying the constraint. Can we locally solve z as a function of x and y near $(0, 1, 1)$? What about x as a function of y and z?

Assuming we can solve $z = z(x, y)$ near $(0, 1, 1)$, and it is a differentiable function, by implicit differentiation we find

$$\begin{cases} 2x + 2z\frac{\partial z}{\partial x} = 0 \\ 2y + 2z\frac{\partial z}{\partial y} = 0 \end{cases},$$

which at $(0, 1, 1)$ gives $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = (0, -1)$.

However, assuming we can solve $x = x(y, z)$ near $(0, 1, 1)$, and it is a differentiable function, by implicit differentiation we find

$$\begin{cases} 2x\frac{\partial x}{\partial y} + 2y = 0 \\ 2x\frac{\partial x}{\partial z} + 2z = 0 \end{cases},$$

which at $(0, 1, 1)$ gives the contradiction $0 = 2$. Therefore, we conclude that even though it may be possible to express $x = x(y, z)$ near $(0, 1, 1)$, such a function cannot be differentiable.

Setting $g(x, y, z) = x^2 + y^2 + z^2$ so that our constraint is $g = 2$, we find that at $(0, 1, 1)$:

$$\frac{\partial g}{\partial z}(0, 1, 1) = 2 \neq 0,$$
$$\frac{\partial g}{\partial x}(0, 1, 1) = 0.$$

More generally, given a constraint $F(x,y,z)=c$, if $z=z(x,y)$, then by implicit differentiation

$$\begin{cases} \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \\ \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0 \end{cases}. \tag{6.20}$$

If $F(\alpha,\beta,\gamma)=c$ and $\frac{\partial F}{\partial z}(\alpha,\beta,\gamma)\neq 0$, then (6.20) has a solution. Therefore $z=z(x,y)$ may exist and be differentiable with

$$\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)|_{(\alpha,\beta,\gamma)}=-\left(\frac{\partial F}{\partial x}(\alpha,\beta,\gamma)\right)^{-1}\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial x}\right)|_{(\alpha,\beta,\gamma)}.$$

Example 144. Consider the curve in $\mathbb{R}^3$ given by the pair of constraints

$$\begin{cases} x^2+y^2+z^2=2, \\ x+z=1. \end{cases}.$$

Locally about the point $(0,1,1)$ is the curve really a one-dimensional curve? That is, near $(0,1,1)$, can we express $y=y(x)$ and $z=z(x)$?

If we assume so, then by implicit differentiation

$$\begin{cases} 2x+2y\frac{dy}{dx}+2z\frac{dz}{dx}=0, \\ 1+\frac{dz}{dx}=0. \end{cases}$$

We can rewrite this linear system of equations in the form

$$\begin{pmatrix} 2y & 2z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix}=\begin{pmatrix} -2x \\ -1 \end{pmatrix}.$$

At the point $(0,1,1)$, this equation becomes

$$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix}=\begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

which has solution

$$\begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix}=\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

More generally, given a pair of constraints $F_i(x,y,z)=c_i$ (for $i=1,2$), if $y=y(x)$ and $z=z(x)$, then by implicit differentiation

$$\frac{\partial F_i}{\partial x}+\frac{\partial F_i}{\partial y}\frac{dy}{dx}+\frac{\partial F_i}{\partial z}\frac{dz}{dx}=0, \quad (\text{for } i=1,2),$$

which can be rewritten in the form

$$\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}\begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix}+\begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{pmatrix}=0. \tag{6.21}$$

If $F_i(\alpha,\beta,\gamma) = c_i$ (for $i = 1,2$), and the matrix $\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}|_{(\alpha,\beta,\gamma)}$ is invertible, then (6.21) has the solution

$$\begin{pmatrix} \frac{dy}{dx} \\ \\ \frac{dz}{dx} \end{pmatrix}|_{(\alpha,\beta,\gamma)} = -\begin{pmatrix} \frac{\partial F_1}{\partial y}(\alpha,\beta,\gamma) & \frac{\partial F_1}{\partial z}(\alpha,\beta,\gamma) \\ \\ \frac{\partial F_2}{\partial y}(\alpha,\beta,\gamma) & \frac{\partial F_2}{\partial z}(\alpha,\beta,\gamma) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \\ \frac{\partial F_2}{\partial x} \end{pmatrix}|_{(\alpha,\beta,\gamma)}.$$

Suppose we are given $n+k$-variables and k-equations

$$\begin{cases} F_1(x_1,\ldots,x_n,y_1,\ldots,y_k) = c_1, \\ \vdots \\ F_k(x_1,\ldots,x_n,y_1,\ldots,y_k) = c_k, \end{cases}$$

When can we (locally) express the variables $y_1,\ldots,y_k$ as functions of $x_1,\ldots,x_n$?

Theorem 145 (Implicit Function Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open and $F : \Omega \to \mathbb{R}^k$ be a C^1-function. Denote $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $y = (y_1,\ldots,y_k) \in \mathbb{R}^k$, and*

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ \vdots \\ F_k(x,y) \end{pmatrix} = \begin{pmatrix} F_1(x_1,\ldots,x_n,y_1,\ldots,y_k) \\ \vdots \\ F_k(x_1,\ldots,x_n,y_1,\ldots,y_k) \end{pmatrix}.$$

Suppose $(a,b) \in \Omega$ is such that $F(a,b) = \lambda \in \mathbb{R}^k$, and that the $k\times k$ matrix

$$\frac{\partial F}{\partial y}(a,b) = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a,b) & \cdots & \frac{\partial F_1}{\partial y_k}(a,b) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial y_1}(a,b) & \cdots & \frac{\partial F_k}{\partial y_k}(a,b) \end{pmatrix},$$

is invertible. Then, there exists open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^k$ with $a \in U$ and $b \in V$, and a unique function $\varphi : U \to V$ such that $\varphi(a) = b$ and

$$F(x,\varphi(x)) = \lambda, \quad \forall x \in U.$$

Moreover, φ is a C^1 function with Jacobian matrix

$$\underbrace{\left(\frac{\partial \varphi}{\partial x}\right)}_{k\times n} = -\underbrace{\left(\frac{\partial F}{\partial y}\right)^{-1}}_{k\times k} \cdot \underbrace{\left(\frac{\partial F}{\partial x}\right)}_{k\times n}, \quad \forall x \in U.$$

Written in full, this is the equation

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1}(x) & \cdots & \frac{\partial \varphi_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_k}{\partial x_1}(x) & \cdots & \frac{\partial \varphi_k}{\partial x_n}(x) \end{pmatrix} = -\begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x,\varphi(x)) & \cdots & \frac{\partial F_1}{\partial y_k}(x,\varphi(x)) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial y_1}(x,\varphi(x)) & \cdots & \frac{\partial F_k}{\partial y_k}(x,\varphi(x)) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x,\varphi(x)) & \cdots & \frac{\partial F_1}{\partial x_n}(x,\varphi(x)) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1}(x,\varphi(x)) & \cdots & \frac{\partial F_k}{\partial x_n}(x,\varphi(x)) \end{pmatrix}.$$

Example 146 (The special case $k = 1$)**.** $F : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$, with $F(x, y) = F(x_1, \dots, x_n, y)$. That is, there is a single constraint $F(x, y) = c \in \mathbb{R}$. Suppose $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $(a, b) \in \Omega$ with $F(a, b) = c$. The Implicit Function Theorem tells you that if $\frac{\partial F}{\partial y}(a, b) \neq 0$, then we can find a differentiable function $y = y(x_1, \dots, x_n)$ locally near $x = a$, with $y(a) = b$, solving the constraint

$$F(x_1, \dots, x_n, y(x_1, \dots, x_n)) = c.$$

Example 147 (The special case $n = 1$, $k = 2$)**.** $F : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^2$, with $F(x, y, z) = F(x, y_1, y_2)$. Given a pair of constraints $F(x, y, z) = c \in \mathbb{R}^2$, suppose $a \in \mathbb{R}$ and $b \in \mathbb{R}^2$ is such that $(a, b) \in \Omega$ with $F(a, b) = c$. The Implicit Function Theorem tells you that if the matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a, b) & \frac{\partial F_1}{\partial y_2}(a, b) \\ \frac{\partial F_2}{\partial y_1}(a, b) & \frac{\partial F_2}{\partial y_2}(a, b) \end{pmatrix},$$

is invertible, then we can find differentiable functions $y_1 = y_1(x), y_2 = y_2(x)$ locally near $x = a$, with $(y_1(a), y_2(a)) = b$, solving the constraint

$$F(x, y_1(x), y_2(x)) = c.$$

Remark. *We write $F = F(x, y)$ and solve y as a function of x, but in fact, the ordering of the variables is irrelevant; we just need to check the invertibility of the appropriate submatrix of the Jacobian matrix $DF(x, y)$.*

Example 148. Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8, \\ x + 4y + 3z = 18. \end{cases}$$

Near $(2, 1, 4)$, can we locally solve for two of the variables as functions of the third?

Letting $F(x, y, z) = (xz + \sin(yz - x^2), x + 4y + 3z)$, we find that

$$DF(x, y, z) = \begin{pmatrix} z - 2x\cos(yz - x^2) & z\cos(yz - x^2) & x + y\cos(yz - x^2) \\ 1 & 4 & 3 \end{pmatrix},$$

and hence

$$DF(2, 1, 4) = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{pmatrix}.$$

By looking at the invertiblilty of 2×2 submatrices we can apply the Implicit Function Theorem.

- $\begin{vmatrix} 0 & 4 \\ 1 & 4 \end{vmatrix} = -4 \neq 0$, and hence by the Implicit Function Theorem, x and y can be expressed as functions of z near $(2, 1, 4)$.
- $\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} = -3 \neq 0$, and hence by the Implicit Function Theorem, x and z can be expressed as functions of y near $(2, 1, 4)$.
- $\begin{vmatrix} 4 & 3 \\ 4 & 3 \end{vmatrix} = 0$, and so no conclusion can be drawn from the Implicit Function Theorem in regards to whether y and z can be expressed as functions of x near $(2, 1, 4)$.

6.3 Inverse Function Theorem

Theorem 149 (Inverse Function Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ be a C^1 function, and $f(a) = b$. Suppose $Df(a)$ is invertible (as an $n \times n$ matrix). Then there exists open sets $U, V \subseteq \mathbb{R}^n$ with $a \in U$ and $b \in V$, and a unique function $g : V \to U$ with $g(b) = a$ such that*

$$\begin{aligned} g \circ f(y) &= y, \quad \forall y \in U, \\ f \circ g(x) &= x, \quad \forall x \in V. \end{aligned}$$

That is g is a local inverse to f. Moreover, g is also a C^1 function with

$$Dg(x) = Df(g(x))^{-1}, \quad \forall x \in V.$$

We note that the Inverse Function Theorem follows from the Implicit Function Theorem:

That is, define $F : f(\Omega) \times \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $F(x, y) = f(y) - x$. Then $F(b, a) = 0$ and $\frac{\partial F}{\partial y}|_{(a,b)} = Df(a)$ is invertible. So by the Implicit Function Theorem, $y = g(x)$ locally with

$$0 = F(x, g(x)) = f \circ g(x) - x.$$

More precisely, $y = g(x)$ is precisely the local inverse of f as claimed in the Inverse Function Theorem.

However, the Inverse Function Theorem is not just a special case of the Implicit Function Theorem, the two theorems are actually equivalent.

Indeed, suppose the Inverse Function Theorem holds and let $F : \Omega \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a C^1 function with $\frac{\partial F}{\partial y}|_{(a,b)}$ be invertible with $F(a, b) = c$. Define $G : \Omega \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ by $G(x, y) = (x, F(x, y))$. Then

$$DG(a, b) = \begin{pmatrix} I_n & 0 \\ \frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \end{pmatrix}.$$

Since $\det(DG(a, b)) = \det \frac{\partial F}{\partial y}(a, b) \neq 0$, $DG(a, b)$ is invertible. Thus, by the Inverse Function Theorem, there exists a local inverse G^{-1}. More precisely, for (x, y) near (a, b) we have

$$\begin{aligned} (x, y) &= G \circ G^{-1}(x, y) \\ &= G(G_1^{-1}(x, y), G_2^{-1}(x, y)) \\ &= (G_1^{-1}(x, y), F(G_1^{-1}(x, y), G_2^{-1}(x, y))). \end{aligned}$$

In particular $x = G_1^{-1}(x, y)$ and $y = F(x, G_2^{-1}(x, y))$. Setting $\varphi(x) = G_2^{-1}(x, c)$, we see that

$$F(x, \varphi(x)) = F(x, G_2^{-1}(x, c)) = c,$$

and φ is precisely the desired function as claimed in the Implicit Function Theorem.

Remark. *For a full proof of the Implicit and Inverse Function Theorems see Math3060.*

Example 150. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the C^1 function $f(x, y) = (x^2 - y^2, 2xy)$. Since $f(-x, -y) = f(x, y)$, f is not injective and so cannot have a global inverse. However, we have that

$$Df(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix},$$

which has determinant $4(x^2 + y^2)$. So $Df(x, y)$ is invertible iff $(x, y) \neq (0, 0)$. Therefore, by the Inverse Function Theorem, f is locally invertible about every point other than the origin.

Suppose $g(u, v)$ is such a local inverse of $f(x, y)$ near the point $(x, y) = (1, -1)$. Then, we find that $g(0, -2) = (1, -1)$ with

$$Dg(0, -2) = Df(1, -1)^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In both theorems, we assume a certain Jacobian matrix is invertible. Without this assumption, the theorems are inconclusive, and the existence of a local implicit or inverse function is unknown.