3.2 Invertibility and row operations.

0. *Assumed background.*

   - What has been covered in Topics 1-2, especially:—
     - 1.7 *Row operations on matrices.*
     - 1.8 *Row operations and matrix multiplication.*
   - 3.1 *Invertible matrices.*

   Abstract. We introduce:—

   - a necessary and sufficient condition for invertibility in terms of row operations,
   - how to determine, for each given square matrix, whether it is invertible or not, and to write down a matrix inverse when it is invertible, in a systematic and methodical way.

1. Recall the definition for the notion of *invertibility* for square matrix:—

   Definition. (Invertibility for square matrix, and matrix inverse.)

   Suppose A is a $(p \times p)$-square matrix.

   Then we say that A is **invertible** *if and only if there is some $(p \times p)$-square matrix B such that B is both a left inverse and a right inverse of A.*

   Such a matrix B is called a **matrix inverse** *of A.*

2. Earlier we have made a claim that we have not justified:—

   $(\star)$ *Any given square matrix has both left and right inverses, or neither.*

   This claim is non-trivial. Logically equivalent to $(\star)$ is the equally non-trivial claim:

   $(\star')$ *It is impossible for a square matrix to have only a left/right inverse, but not a right/left inverse respectively.*

   We now start to pave the way to justify the claim $(\star)$.

   This starts with two questions:—

   Question (1). *How to determine, for each given square matrix, whether it is invertible or not, and to write down a matrix inverse when it is invertible, in a systematic and methodical way?*

   Question (2). *How is invertibility of square matrices related with what we have learnt already on row operations (and/or other things, such as systems of linear equations, linear combinations, linear dependence and linear independence)?*

3. To obtain some hints for the answers to the questions above, perhaps we first ask the question below:—

   Question (0). *Which square matrices are already known to be invertible because we know how to write down their matrix inverses explicitly?*

   In fact we know a lot of such square matrices, from our study in row operations and row operation matrices. We recall this result, labelled Theorem (†) here:—

   Theorem (†).

   Suppose ρ is a row operation on matrices with p rows, and $\widetilde{\rho}$ is the 'reverse row operation'.

   (Denote by $M[\rho], M[\widetilde{\rho}]$ the $(p \times p)$-square matrices which are the row-operation matrices associated with $\rho, \widetilde{\rho}$ respectively.)

   Then $M[\widetilde{\rho}]M[\rho] = I_p$ and $M[\rho]M[\widetilde{\rho}] = I_p$.

   Remark. For reference, the 'formulae' for the row-operation matrices of 'reverse row operations' for row operations of various types are explicitly described in the table below:—

Row operation ρ on matrices with p rows.	Row-operation matrix $M[\rho]$.	'Reverse row operation' $\widetilde{\rho}$ for ρ.	Row-operation matrix $M[\widetilde{\rho}]$.
$\alpha R_i + R_k$.	$I_p + \alpha E_{k,i}^{p,p}$.	$-\alpha R_i + R_k$.	$I_p - \alpha E_{k,i}^{p,p}$.
βR_k.	$I_p + (\beta - 1) E_{k,k}^{p,p}$.	$(1/\beta) R_k$.	$I_p + (1/\beta - 1) E_{k,k}^{p,p}$.
$R_i \leftrightarrow R_k$.	$I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p}$.	$R_i \leftrightarrow R_k$.	$I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p}$.

4. In terms of the definition for the notions of invertibility and matrix inverse, Theorem (†) can be re-formulated as:—

   Theorem (1).

   Suppose ρ is a row operation on matrices with p rows, and $\widetilde{\rho}$ is the 'reverse row operation'.

   Then the row-operation matrices $M[\rho], M[\widetilde{\rho}]$ associated with $\rho, \widetilde{\rho}$ respectively are invertible, and are matrix inverses of each other:

$$M[\rho]^{-1} = M[\widetilde{\rho}] \quad \text{and} \quad M[\widetilde{\rho}]^{-1} = M[\rho].$$

5. Recall the result below, labelled Theorem (‡), which has been proved earlier:—

   Theorem (‡).

   Let A, B be $(p \times p)$-square matrices.

   Suppose A, B are invertible.

   Then the product AB is invertible with matrix inverse given by $(AB)^{-1} = B^{-1}A^{-1}$.

6. A consequence of Theorem (1) and Theorem (‡) (with an application of mathematical induction) is Theorem (2):—

   Theorem (2).

   All products of row-operation matrices are invertible.

7. Thinking in terms of row operations, we can translate the result described in Theorem (2) into:—

   - *Whenever a square matrix is the resultant of the application of some sequence of row operations on the identity matrix, that square matrix is invertible.*

   So we suddenly know of much more examples of invertible matrices.

   Below are some examples of invertible matrices arising from specific types of row operations.

8. **Example (1). (Diagonal matrices with non-zero diagonal entries (of small size) as products of row-operation matrices.)**

   Let $\alpha, \alpha_2, \alpha_3, \alpha_4$ be non-zero numbers.

   The diagonal matrix

$$\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}$$

   is invertible.

   Seen from the point of view of row operations, $\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the resultant of the applications on I_4 of the sequences of row operations

$$I_4 \xrightarrow{\alpha_1 R_1} \xrightarrow{\alpha_2 R_2} \xrightarrow{\alpha_3 R_3} \xrightarrow{\alpha_4 R_4} \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

   When we think in terms of row-operation matrices, we obtain the equality between $\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and a product of several row-operation matrices:

$$\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Remark. We know that every diagonal matrix whose diagonal entries are non-zero. One way to see this is to think of such a matrix as the resultant of a sequence of row operations of the type 'multiplying a non-zero scalar to a row'.

9. **Example (2). (Upper uni-triangular matrices (of small size) as products of row-operation matrices.)**

   Let $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$ be numbers.

   The matrix

$$U(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}) = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \\ 0 & 0 & 1 & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   is invertible, as $U(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34})$ is the resultant of the applications on I_4 of the sequences of row operations

$$I_4 \xrightarrow{\alpha_{12}R_2+R_1} \xrightarrow{\alpha_{13}R_3+R_1} \xrightarrow{\alpha_{23}R_3+R_2} \xrightarrow{\alpha_{14}R_4+R_1} \xrightarrow{\alpha_{24}R_4+R_2} \xrightarrow{\alpha_{34}R_4+R_3} U(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}).$$

When we think in terms of row-operation matrices, we obtain the equality between $U(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34})$ and a product of several row-operation matrices:

$$\begin{aligned} & U(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}) \\ = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \alpha_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Remark. What we have seen is an upper uni-triangular matrix of small size.

In general, any $(p \times p)$-square matrix which reads

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1,p-2} & \alpha_{1,p-1} & \alpha_{1p} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2,p-2} & \alpha_{2,p-1} & \alpha_{2p} \\ 0 & 0 & \alpha_{33} & \ddots & & \alpha_{3,p-1} & \alpha_{3p} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \alpha_{p-2,p-2} & \alpha_{p-2,p-1} & \alpha_{p-2,p} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{p-1,p-1} & \alpha_{p-1,p} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_{pp} \end{bmatrix}$$

(in the (i, j)-th entry is 0 whenever $i > j$) is called an **upper triangular matrix**.

If all the diagonal entries $\alpha_{11}, \alpha_{22}, \cdots, \alpha_{pp}$ are equal to 1, such a matrix is called an **upper uni-triangular matrix**.

Every upper uni-triangular matrix is invertible, as it is the resultant on the identity matrix of a sequence of row operations of the type ‘adding a scalar multiple of one row to another’.

Further remark. Analogous to upper triangular matrices and upper uni-triangular matrices are **lower triangular matrices** and **lower uni-triangular matrices**. (Formulate their respective definitions as exercises.) Every lower uni-triangular matrix is invertible, being the resultant on the identity matrix of a sequence of row operations of the type ‘adding a scalar multiple of one row to another’.

10. **Example (3). (Permutation matrices (of small size) as products of row-operation matrices.)**

Let $M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The matrix M is invertible, as it is the resultant of the application on I_5 of the sequence of row operations

$$I_5 \xrightarrow{R_1 \leftrightarrow R_5} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{R_3 \leftrightarrow R_4} \xrightarrow{R_4 \leftrightarrow R_5} M$$

When we think in terms of row-operation matrices, we obtain the equality between M and a product of several row-operation matrices:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remark. What we have seen is a permutation matrix of small size.

Any $(p \times p)$-matrix in which there is exactly one 1 in each row and each column, and every other entry is 0 is called a **permutation matrix**.

- These are all the (2×2)-permutation matrices:—

  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- These are all the (3×3)-permutation matrices:—

  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

- There are twenty-four (4×4)-permutation matrices, and one hundred and twenty (5×5)-permutation matrices. Et cetera.

Every permutation matrix is invertible, as it is the resultant on the identity matrix of a sequence of row operations of the type 'interchanging two rows'. (This can be proved with the method of mathematical induction.)

11. Theorem (2) and Example (1), Example (2), Example (3) suggest another question:

    Question (3). Is there any invertible matrix other than those which are resultant from the application on the identity matrix of various sequences of row operations?

12. All the questions we have posed above are comprehensively answered by the result below.

    Theorem (3). (Re-formulation of invertibility in terms of row operations.)

    Suppose A is a $(p \times p)$-square matrix. Then the statements below are logically equivalent:—

    (a) *A is invertible.*

    (b) *A is row-equivalent to I_p.*

    (c) *A is a product of $(p \times p)$-row-operation matrices.*

    Now suppose any one of the above holds (and hence all hold). Then the statements below hold:—

    (α) *$[\ A \mid I_p\ ]$ is row-equivalent to $[\ I_p \mid A^{-1}\ ]$.*

    (β) *I_p is the only reduced row-echelon form which is row-equivalent to A.*

    Remark. The proof of Theorem (3) is postponed. In fact, we will later state a 'fuller version' of Theorem (3), and give the full argument. For the moment we take its validity for granted, and see:—

    - How Theorem (3) is applied to determine, for each given square matrix, whether it is invertible or not, and to write down a matrix inverse when it is invertible, in a systematic and methodical way.

13. **'Algorithm' associated to Theorem (3).**

    Let A be a $(p \times p)$-square matrix.

    We are going to determine whether A is invertible, and to obtain the matrix inverse A^{-1} when it is invertible:

    Step (1). Form the $(p \times 2p)$-matrix $C = [\ A \mid I_p\ ]$.

    Step (2). Obtain some row-echelon form $C^\sharp$ which is row-equivalent to C, and identify the $(p \times p)$-square matrices $A^\sharp, B^\sharp$ for which the equality $C^\sharp = \left[\ A^\sharp \mid B^\sharp\ \right]$ holds.

    (Note that $A^\sharp$ is a row-echelon form, because $C^\sharp$ is a row-echelon form.)

    Step (3). Inspect the row-echelon form $A^\sharp$, and ask:—

    Is every column of $A^\sharp$ a pivot column?

    - If *no*, then conclude that A is not invertible.
    - If *yes*, then conclude that A is invertible.
      To obtain the matrix inverse of A, go to Step (4).

    Step (4). Obtain some reduced row-echelon form C' which is row-equivalent to $C^\sharp$, and identify the $(p \times p)$-square matrix B' for which $C' = [\ I_p \mid B'\ ]$.

    Conclude that $A^{-1} = B'$.

    (As a bonus we also obtain a 'factorization' of A and A^{-1} as products of row-operation matrices, running through all four steps.)

14. **Example (4). (Illustrations on the use of row operations for determining whether a square matrix is invertible and for computing matrix inverse.)**

    (a) *Let* $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix}$.

    We want to determine whether A is invertible, and to find the matrix inverse of A if A is invertible.

    Define $C = [\ A \mid I_3\ ]$.

Obtain from C a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C = \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -2 & -4 & -1 & 1 & 0 \\ 0 & -4 & -11 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -2 & -4 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right] = C^\sharp = [\ A^\sharp \mid B^\sharp\ ]$$

in which $A^\sharp = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 0 & -3 \end{bmatrix}$, $B^\sharp = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$.

Note that every column in the row-echelon form $A^\sharp$ is pivot column. Then A is invertible.

(What is done next is to obtain the matrix inverse of A.)

We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -2 & -4 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/2 & 3/2 & 0 \\ 0 & 1 & 2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right]$$

$$\xrightarrow{1R_3+R_1} \xrightarrow{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & 13/6 & -1/3 \\ 0 & 1 & 0 & -1/6 & -11/6 & 2/3 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right] = C' = [\ I_3 \mid A^{-1}\ ]$$

Hence $A^{-1} = \begin{bmatrix} -1/6 & 13/6 & -1/3 \\ -1/6 & -11/6 & 2/3 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$.

As a bonus we obtain these factorizations:—

$$A = G_1G_2G_3G_4G_5G_6G_7G_8, \quad \text{and} \quad A^{-1} = {G_8}^{-1}{G_7}^{-1}{G_6}^{-1}{G_5}^{-1}{G_4}^{-1}{G_3}^{-1}{G_2}^{-1}{G_1}^{-1},$$

in which

$$\begin{cases} G_1 = M[1R_1+R_2], & {G_1}^{-1} = M[-1R_1+R_2] \\ G_2 = M[3R_1+R_3], & {G_2}^{-1} = M[-3R_1+R_3] \\ G_3 = M[1R_2+R_3], & {G_3}^{-1} = M[-1R_2+R_3] \\ G_4 = M[-2R_2], & {G_4}^{-1} = M[-\frac{1}{2}R_2] \\ G_5 = M[-3R_3], & {G_5}^{-1} = M[-\frac{1}{3}R_3] \\ G_6 = M[3R_2+R_1], & {G_6}^{-1} = M[-3R_2+R_1] \\ G_7 = M[-1R_3+R_1], & {G_7}^{-1} = M[1R_3+R_1] \\ G_8 = M[2R_3+R_2], & {G_8}^{-1} = M[-2R_3+R_2] \end{cases}$$

(b) *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 7 & 15 \\ 3 & 8 & 10 & 20 \\ 4 & 13 & 17 & 38 \end{bmatrix}$.

We want to determine whether A is invertible, and to find the matrix inverse of A if A is invertible.

Define $C = [\ A \mid I_4\ ]$.

Obtain from C a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C = \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 6 & 1 & 0 & 0 & 0 \\ 2 & 5 & 7 & 15 & 0 & 1 & 0 & 0 \\ 3 & 8 & 10 & 20 & 0 & 0 & 1 & 0 \\ 4 & 13 & 17 & 38 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \xrightarrow{-3R_1+R_3} \xrightarrow{-4R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 5 & 5 & 14 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \xrightarrow{-5R_2+R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -4 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 6 & -5 & 0 & 1 \end{array}\right] = C^\sharp = [\ A^\sharp \mid B^\sharp\ ]$$

in which $A^\sharp = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $B^\sharp = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 6 & -5 & 0 & 1 \end{bmatrix}$.

Note that every column in the row-echelon form $A^\sharp$ is pivot column. Then A is invertible.

(What is done next is to obtain the matrix inverse of A.)

We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -4 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 6 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3}\xrightarrow{-1R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 5 & 0 & -1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 5 & -2 & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 5 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_1}\xrightarrow{-1R_3+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 5 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{4R_4+R_1}\xrightarrow{1R_4+R_2}\xrightarrow{-4R_4+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -18 & 16 & 1 & -4 \\ 0 & 1 & 0 & 0 & -7 & 4 & 1 & -1 \\ 0 & 0 & 1 & 0 & 23 & -18 & -1 & 4 \\ 0 & 0 & 0 & 1 & -6 & 5 & 0 & -1 \end{array}\right] = C' = [\ I_4 \mid A^{-1}\ ]$$

Hence $A^{-1} = \begin{bmatrix} -18 & 16 & 1 & -4 \\ -7 & 4 & 1 & -1 \\ 23 & -18 & -1 & 4 \\ -6 & 5 & 0 & -1 \end{bmatrix}$.

As a bonus we obtain these factorizations:—

$$A = G_1G_2G_3G_4G_5G_6G_7G_8G_9G_{10}G_{11}G_{12}G_{13}$$

and

$$A^{-1} = {G_{13}}^{-1}{G_{12}}^{-1}{G_{11}}^{-1}{G_{10}}^{-1}{G_9}^{-1}{G_8}^{-1}{G_7}^{-1}{G_6}^{-1}{G_5}^{-1}{G_4}^{-1}{G_3}^{-1}{G_2}^{-1}{G_1}^{-1},$$

in which

$$\left\{\begin{array}{lllllll} G_1 & = & M[2R_1+R_2], & & {G_1}^{-1} & = & M[-2R_1+R_2] \\ G_2 & = & M[3R_1+R_3], & & {G_2}^{-1} & = & M[-3R_1+R_3] \\ G_3 & = & M[4R_1+R_4], & & {G_3}^{-1} & = & M[-4R_1+R_4] \\ G_4 & = & M[2R_2+R_3], & & {G_4}^{-1} & = & M[-2R_2+R_3] \\ G_5 & = & M[5R_2+R_4], & & {G_5}^{-1} & = & M[-5R_2+R_4] \\ G_6 & = & M[-1R_3], & & {G_6}^{-1} & = & M[-1R_3] \\ G_7 & = & M[-1R_4], & & {G_7}^{-1} & = & M[-1R_4] \\ G_8 & = & M[2R_2+R_1], & & {G_8}^{-1} & = & M[-2R_2+R_1] \\ G_9 & = & M[1R_3+R_1], & & {G_9}^{-1} & = & M[-1R_3+R_1] \\ G_{10} & = & M[1R_3+R_2], & & {G_{10}}^{-1} & = & M[-1R_3+R_2] \\ G_{11} & = & M[-4R_4+R_1], & & {G_{11}}^{-1} & = & M[4R_4+R_1] \\ G_{12} & = & M[-1R_4+R_2], & & {G_{12}}^{-1} & = & M[1R_4+R_2] \\ G_{13} & = & M[4R_4+R_3], & & {G_{13}}^{-1} & = & M[-4R_4+R_3] \end{array}\right.$$

(c) *Let* $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 14 \\ -1 & 3 & 7 \end{bmatrix}$.

We want to determine whether A is invertible, and to find the matrix inverse of A if A is invertible.

Define $C = [\ A \mid I_3\ ]$.

Obtain from C a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C = \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 2 & 8 & 14 & 0 & 1 & 0 \\ -1 & 3 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2}\xrightarrow{1R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 1 & 0 \\ 0 & 6 & 12 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & 7 & -3 & 1 \end{array}\right] = C^\sharp = [\ A^\sharp \mid B^\sharp\ ]$$

in which $A^\sharp = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $B^\sharp = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -3 & 1 \end{bmatrix}$.

Note that some column in the row-echelon form $A^\sharp$, such as the 3-rd column, fails to be a pivot column. Then A is not invertible.

(d) *Let* $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 14 & 24 & 36 \\ 1 & 9 & 17 & 28 \\ 2 & 4 & 6 & 12 \end{bmatrix}$.

We want to determine whether A is invertible, and to find the matrix inverse of A if A is invertible.

Define $C = [\ A \mid I_4\ ]$.

Obtain from C a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C = \left[\begin{array}{cccc|cccc} 1 & 3 & 5 & 7 & 1 & 0 & 0 & 0 \\ 4 & 14 & 24 & 36 & 0 & 1 & 0 & 0 \\ 1 & 9 & 17 & 28 & 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 12 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-4R_1+R_2} \xrightarrow{-1R_1+R_3} \xrightarrow{-2R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 3 & 5 & 7 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 6 & 12 & 21 & -1 & 0 & 1 & 0 \\ 0 & -2 & -4 & -2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3} \xrightarrow{1R_2+R_4} \left[\begin{array}{cccc|cccc} 1 & 3 & 5 & 7 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 11 & -3 & 1 & 0 \\ 0 & 0 & 0 & 6 & -6 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 3 & 5 & 7 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 11 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 16 & -5 & 2 & 1 \end{array}\right] = C^\sharp = [\ A^\sharp \mid B^\sharp\ ]$$

in which $A^\sharp = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B^\sharp = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 11 & -3 & 1 & 0 \\ 16 & -5 & 2 & 1 \end{bmatrix}$.

Note that some column in the row-echelon form $A^\sharp$, such as the 3-rd column, fails to be a pivot column. Then A is not invertible.